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Electromagnetic Structure of the Giant Dipole Resonance*

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The inelastic transverse form factor for electromagnetic excitation of the giant resonance is discussed and calculated on the basis of several different models. Particular attention is paid to the Brown theory of the giant resonance, which is reformulated here in such a way that no free parameters are left over, i.e., all parameters are determined from other experiments. Numerical calculations are carried out for C^{12} and compared with the 180° electron scattering experiments and with photoabsorption data. It is found that the Brown theory successfully predicts the dip in the squared form factor and the shift in the resonance energy as the momentum transfer is changed, both of which are observed experimentally, and neither of which is predicted by the collective models for the charge motion of the giant resonance. More detailed comparison with the experimental data allows us to distinguish between different versions of the Brown theory, and it is found that the no-free-parameter results are consistent with all the experimental data considered. Results for all the models considered are presented and discussed.

I. INTRODUCTION

THE nature of the giant dipole resonance has long been an interesting question in nuclear physics. Experimentally, it is known that the giant resonance exhausts most of the sum rule for electric dipole matrix elements. It is the dominant element of structure in low-energy nuclear physics and this is why it has received so much attention both experimentally and theoretically. Goldhaber and Teller[1] proposed that the giant resonance represents an oscillation of the neutrons against the protons in the nucleus. Such a model gives an electric dipole matrix element equal to the sum rule value, but is incapable of explaining the more detailed features of the giant resonance, such as its splitting into several peaks.[2,3] Wilkinson[4] realized that if one sums the individual particle transition strengths in an oscillator potential, or oscillator potential with a spin-orbit force, then one again finds a transition strength which exhausts the dipole sum rule. The difficulty in this case is that stripping experiments, and calculation of configuration energies from neighboring nuclei, show that these single-particle states lie too low in energy to explain the appearance of the giant dipole resonance in the region 15–25 MeV. Brown and his collaborators here pointed out that if one regarded the resonance state as made up of a linear combination of particle-hole states, then the residual particle-hole interaction being of opposite sign to the particle-particle interaction, and hence *repulsive* would tend to drive the resonance states up in energy. Several calculations of energy levels and relative dipole strengths in C^{12} and O^{16} have been carried out on the basis of this model by Brown and his co-workers[5–8] and an intensive application of the particle-hole idea to other states and other nuclei has been undertaken by Gillet.[9] A detailed intermediate coupling calculation on odd-parity states in light nuclei

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[2] W. C. Barber and W. R. Dodge, Phys. Rev. **127**, 1746 (1962). See also W. C. Barber and W. R. Dodge, *Contributions to the Karlsruhe Photonuclear Conference 1960* (Erstes Physikalisches Institut der Universität Heidelberg, Heidelberg, 1961), p. A7.
[3] N. W. Tanner, G. C. Thomas, and E. D. Earle, *Proceedings of the Rutherford Jubilee International Conference, Manchester, 1961*, edited by J. B. Birks (Academic Press Inc., New York, 1961), paper C2/30.
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[5] G. E. Brown and M. Bolsterli, Phys. Rev. Letters **3**, 472 (1959).
[6] G. E. Brown, L. Castillejo, and J. A. Evans, Nucl. Phys. **22**, 1 (1961).
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[8] G. E. Brown, L. Castillejo, and J. A. Evans, *Contributions to the Karlsruhe Photonuclear Conference 1960* (Erstes Physikalisches Institut der Universität Heidelberg, Heidelberg, 1961), p. B4.
[9] V. Gillet, thesis, Universite de Paris, Saclay, 1962, C.E.A. Report No. 2177 (unpublished).

carried out earlier by Elliott and Flowers[10] also shows that the residual interactions raise the energy of the dipole states. The results for the $T=1, J^\pi=1^-$ states in O^{16} turn out to be very similar in the two calculations.

One of the most powerful tools available for elucidating nuclear structure is inelastic electron scattering. The interaction of electrons with nucleons is well known and therefore, by varying the momentum transferred to the nucleus at a fixed energy loss, one can map out the Fourier transform of the transition charge and current densities and hence the charge and current densities themselves. This gives one much more detailed information about the nature of a state than does just the total photon absorption width, which is just one number, for example. In this paper we shall study the question of what one can learn about the nature of the giant dipole resonance through the use of inelastic electron scattering. In Sec. II we give a brief review of inelastic electron scattering and also discuss the relation to photoabsorption. In Sec. III we give a discussion of the Brown theory in terms of wave functions and a "Tamm-Dancoff' approximation. While this discussion is merely a *reformulation* of the Brown theory, it does help one to understand a little more clearly, we feel, such things as the nature of the approximations involved, what force is to be used, how one is to experimentally identify the unperturbed configuration energies and the relation to the intermediate-coupling calculations. It also allows one to easily keep track of all the phases involved, which in a calculation such as this is a nontrivial matter.

Section III contains a detailed discussion of the calculation of the energy matrix and electron scattering matrix elements in the Brown theory. The electron scattering cross section based on the Goldhaber-Teller model has been calculated in a previous paper.[11] The cross section given by the Steinwedel-Jensen hydrodynamical model is discussed in Appendix A. One of the major conclusions of the present work is that the various models give *completely different* inelastic form factors. In Sec. IV an application to C^{12} is carried out and the results for the transverse electric dipole form factors are compared with the recent results of Goldemberg *et al.*,[11,12] who look at electron scattering in the backward direction with the 75-MeV Mark II linear accelerator at Stanford. A second conclusion of the present paper is that the Brown theory gives very characteristic form factors which are quantitatively in agreement with the experimental results both as to shape and magnitude. Similar results have been obtained for O^{16} and will be discussed in a forthcoming paper.[13] Section V contains a discussion and summary.

II. INELASTIC ELECTRON SCATTERING

The cross section for inelastic electron scattering from the ground state to a discrete excited state is given in Born approximation, with the neglect of nuclear recoil and the electron mass with respect to its energy, by

$$\frac{d\sigma}{d\Omega}(J_f \leftarrow J_i)=\frac{k_2}{k_1}\frac{8\pi\alpha^2}{\Delta^4}\Big[V_L(\theta)\sum_{J=0}^{\infty}\frac{1}{2J_i+1}|(J_f\|M_J(q)\|J_i)|^2 + V_T(\theta)\sum_{J=1}^{\infty}\frac{1}{2J_i+1}(|(J_f\|T_J^{\mathrm{mag}}(q)\|J_i)|^2+|(J_f\|T_J^{\mathrm{el}}(q)\|J_i)|^2)\Big]. \quad (1)$$

$\mathbf{k}_1$ and $\mathbf{k}_2$ are the initial and final electron wave numbers, $\mathbf{q}^2=(\mathbf{k}_2-\mathbf{k}_1)^2$ and $\Delta^2=\mathbf{q}^2-(k_2-k_1)^2$ are the three- and four-momentum transfers and θ is the electron scattering angle. We further have

$$V_L(\theta)=\frac{\Delta^4}{q^4}2k_1k_2\cos^2\frac{\theta}{2},$$
$$V_T(\theta)=\frac{2k_1k_2}{q^2}\sin^2\frac{\theta}{2}\Big[(k_1+k_2)^2-2k_1k_2\cos^2\frac{\theta}{2}\Big]. \quad (2)$$

The multipole operators are given by

$$M_{JM}(q)=\int d\mathbf{x}\rho_N(\mathbf{x})j_J(qx)Y_{JM}(\Omega_x),$$
$$T_{JM}^{\mathrm{el}}(q)=\frac{1}{q}\int d\mathbf{x}[\mathbf{j}_N(\mathbf{x})\cdot(\nabla\times j_J(qx)\mathcal{Y}_{JJ1}^M(\Omega_x))+q^2j_J(qx)\mathcal{Y}_{JJ1}^M(\Omega_x)\cdot\mathbf{u}_N(\mathbf{x})], \quad (3)$$
$$T_{JM}^{\mathrm{mag}}(q)=\int d\mathbf{x}[\mathbf{u}_N(\mathbf{x})\cdot(\nabla\times j_J(qx)\mathcal{Y}_{JJ1}^M(\Omega_x))+j_J(qx)\mathcal{Y}_{JJ1}^M(\Omega_x)\cdot\mathbf{j}_N(\mathbf{x})].$$

[10] J. P. Elliott and B. H. Flowers, Proc. Roy. Soc. (London) **A242**, 57 (1957).
[11] J. Goldemberg, Y. Torizuka, W. C. Barber, and J. D. Walecka, Nucl. Phys. **43**, 242 (1963).
[12] F. H. Lewis, Jr., J. D. Walecka, J. Goldemberg, and W. C. Barber, Phys. Rev. Letters **10**, 493 (1963).
[13] F. H. Lewis, Jr. (to be published).

$e\rho_N(\mathbf{x})$, $e\mathbf{j}_N(\mathbf{x})$, $e\boldsymbol{\mu}_N(\mathbf{x})$ are the nuclear charge, current, and magnetization density operators, and

$$\mathcal{Y}_{JJ1}{}^{M}(\Omega)=\sum_{m,\lambda}(Jm1\lambda|J1JM)Y_{Jm}(\Omega)\mathbf{e}_\lambda$$

are the vector spherical harmonics.[14] The operators $T_{JM}{}^{\mathrm{el}}(q)$ and $T_{JM}{}^{\mathrm{mag}}(q)$ are *exactly the same* operators which describe the emission and absorption of *real photons*, only in that case one has a relation between the momentum transfer and the energy loss, $\hbar c|\mathbf{q}|=\Delta E\equiv E_{fi}$. The total integrated photoabsorption cross section to a discrete level can be given in terms of these same operators by (see Appendix B)

$$\int_{\text{one level}}\sigma_{\mathrm{abs}}(E)dE=(2\pi)^3\alpha\frac{(\hbar c)^2}{E_{fi}}\frac{1}{2J_i+1}\sum_J\left[\left|\left(J_f\left\|T_J{}^{\mathrm{el}}\left(\frac{E_{fi}}{\hbar c}\right)\right\|J_i\right)\right|^2+\left|\left(J_f\left\|T_J{}^{\mathrm{mag}}\left(\frac{E_{fi}}{\hbar c}\right)\right\|J_i\right)\right|^2\right]. \tag{4}$$

We shall be concerned with the reduced matrix elements of the various multipole operators as a function of q and will refer to these as the inelastic form factors. One can thus get *one point* on the transverse form factors from photoabsorption. The transverse and longitudinal form factors can be separated experimentally in electron scattering by doing experiments at fixed q but different θ. In the backward direction one has $V_L(\pi)=0$ so one sees just the transverse form factors.

III. BROWN THEORY OF THE GIANT RESONANCE

In this section we shall describe a procedure which is merely a reformulation of the Brown theory. Hopefully, one can gain some further insight into the theory and into the nature of the approximations involved. The nuclear Hamiltonian is given by

$$H=\sum_{i=1}^{A}T(i)+\sum_{i<j=1}^{A}v(i,j). \tag{5}$$

We will assume that $v(ij)$ is a nonsingular potential which fits low-energy nucleon-nucleon scattering. This, of course, does not allow us to account for nuclear saturation. We will find, however, that most of the matrix elements of the potential involved in our calculation can be identified with energies of neighboring nuclei, which we shall take from experiment. We will thus be left with single particle-hole matrix elements of $v(ij)$ and it is known, at least in doubly magic $+2$ nucleon nuclei,[15,16] that the singular nature of the potential does not drastically change the two-particle energies and wave functions.

We shall deal only with nuclei which have doubly closed j shells. For the ground state of such nuclei we will use the simple shell model wave function (we shall take nuclei with 2 shells, as in C^{12} as an illustration; the results are immediately generalized). We write

$$\Phi_0{}^{n_0l_0j_0}(1,2,\cdots 2N_{j_0})\equiv\frac{1}{((2N_{j_0})!)^{1/2}}\sum_P(-1)^PP[\phi_{j_0\uparrow}(1)\phi_{j_0-1\uparrow}(2)\cdots\phi_{-j_0\uparrow}(N_{j_0})\phi_{j_0\downarrow}(N_{j_0}+1)\cdots\phi_{-j_0\downarrow}(2N_{j_0})], \tag{6}$$

where the indices on the single-particle wave functions are m_{j_0} and m_t, the third components of angular momentum and isotopic spin. We assume the single-particle wave functions are eigenstates of $\mathbf{l}^2$, $\mathbf{s}^2$, $\mathbf{t}^2$, $\mathbf{j}^2$, j_z, and t_z. The single-particle states don't have to be specified any further than this for the purposes of our present discussion. The operator $1/((2N_{j_0})!)^{1/2}\sum_P(-1)^PP$ where $N_{j_0}\equiv 2j_0+1$, is the usual antisymmetrization operator containing a sum over all permutations of the $2N_{j_0}$ particles with a sign equal to the signature of the permutation. This wave function is antisymmetric, normalized, and an eigenstate of $\mathbf{J}^2$, J_z, $\mathbf{T}^2$, T_z all with eigenvalues 0. Similarly

$$\Phi_0{}^{nlj}(1,2,\cdots 2N_j)=\frac{1}{((2N_j)!)^{1/2}}\sum_P(-1)^PP[\phi_{j\uparrow}(1)\cdots\phi_{-j\uparrow}(N_j)\phi_{j\downarrow}(N_j+1)\cdots\phi_{-j\downarrow}(2N_j)]. \tag{7}$$

The ground-state wave function is then taken to be

$$\Psi_0=\Phi_0=\frac{1}{((2N_j)!)^{1/2}}\frac{1}{((2N_{j_0})!)^{1/2}}\frac{1}{(A!)^{1/2}}\sum_P(-1)^PP[\Phi_0{}^{n_0l_0j_0}(1\cdots 2N_{j_0})\Phi_0{}^{nlj}(2N_{j_0}+1\cdots A)]. \tag{8}$$

This wave function is normalized, antisymmetric and has $J=T=0$. We will attempt to describe the giant resonance electric dipole states by taking linear combinations of states of single-particle excitation which have un-

[14] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, 1957). We use Edmonds' notation.
[15] J. F. Dawson, I. Talmi, and J. D. Walecka, Ann. Phys. (N. Y.) **18**, 339 (1962).
[16] J. F. Dawson and J. D. Walecka, Ann. Phys. (N. Y.) **22**, 133 (1963).

perturbed configuration energies (to be identified later) lying closest to the observed value of the giant resonance energy. A single hole in the j_0 shell for example can be written as

$$\Phi_{m_0\lambda_0}{}^{(n_0l_0j_0)^{-1}}(1,\cdots 2N_{j_0}-1)=\frac{(-1)^{\frac{1}{2}-\lambda_0}}{((2N_{j_0}-1)!)^{1/2}}\sum_P(-1)^P P[\phi_{j_0\uparrow}(1)\cdots[\phi_{-m_0-\lambda_0}]\cdots\phi_{-j_0\downarrow}(2N_{j_0}-1)], \tag{9}$$

where the state $\phi_{-m_0-\lambda_0}$ has just been omitted from the product wave function as symbolized by the heavy brackets. This wave function has $\mathbf{J}^2=j_0(j_0+1)$, $J_z=m_0$, $\mathbf{T}^2=\frac{1}{2}\cdot\frac{3}{2}$, $T_z=\lambda_0$, as is easily verified. Note that an extra phase $(-1)^{\frac{1}{2}-\lambda_0}$ must be added to achieve this. A particle-hole state of definite j can now be constructed by taking

$$\Phi_{JM_JTM_T}{}^{(n_1l_1j_1)(n_0l_0j_0)^{-1}}(1,2,\cdots 2N_{j_0})=\frac{1}{((2N_{j_0}-1)!)^{1/2}}\sum_{m_0\lambda_0}\sum_{m_1\lambda_1}(j_1m_1j_0m_0|\,j_1j_0JM_J)$$

$$\times(\tfrac{1}{2}\lambda_1\tfrac{1}{2}\lambda_0|\,\tfrac{1}{2}\,\tfrac{1}{2}TM_T)\frac{1}{((2N_{j_0})!)^{1/2}}\sum_P(-1)^P P[\Phi_{m_0\lambda_0}{}^{(n_0l_0j_0)^{-1}}(1\cdots 2N_{j_0}-1)\phi_{m_1\lambda_1}{}^{n_1l_1j_1}(2N_{j_0})]. \tag{10}$$

Note that we have now defined a definite order of coupling. The total basis wave function describing a given particle-hole state is

$$\mathbf{\Phi}_{JM_JTM_T}{}^{(n_1l_1j_1)(n_0l_0j_0)^{-1}}(1,\cdots A)=\frac{1}{((2N_{j_0})!)^{1/2}}\frac{1}{((2N_j)!)^{1/2}}\frac{1}{(A!)^{1/2}}$$

$$\times\sum_P(-1)^P P[\Phi_{JM_JTM_T}{}^{(n_1l_1j_1)(n_0l_0j_0)^{-1}}(1\cdots 2N_{j_0})\Phi_0{}^{nlj}(2N_{j_0}+1\cdots A)]. \tag{11}$$

To describe the giant resonance, we must have states of odd parity and so the particle states we mix in must have opposite parity from the states in the hole-shell. These states lie fairly high ($\sim$1 oscillator spacing or 15 MeV) in nuclei. We now try to construct an excited-state wave function by taking linear combinations of the basis states (we use the labels 1 for particles or 2 for holes).

$$\Psi_{JM_JTM_T}(1\cdots A)=\sum_{\substack{n_1l_1j_1\\ n_2l_2j_2}}\alpha_{JT}{}^{(n_1l_1j_1)(n_2l_2j_2)^{-1}}\mathbf{\Phi}_{JM_JTM_T}{}^{(n_1l_1j_1)(n_2l_2j_2)^{-1}}(1\cdots A). \tag{12}$$

If we try to solve the Schrödinger equation in this basis, we have

$$H\Psi_{JM_JTM_T}=E\Psi_{JM_JTM_T} \tag{13}$$

and, inserting the above wave functions and using their orthonormality properties, we find

$$\sum_{K'}[(\mathbf{\Phi}^K|H|\mathbf{\Phi}^{K'})-E\delta_{KK'}]\alpha_{JT}{}^{K'}=0, \tag{14}$$

where K stands for a pair $(n_1l_1j_1)(n_2l_2j_2)^{-1}$. Diagonalization of this set of linear equations gives us the energy eigenvalues E and sets of coefficients α^K. We assume that the two-particle potential has the form

$$v=v^{(1)}(r_{12})+v^{(2)}(r_{12})\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2+v^{(3)}(r_{12})\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2+v^{(4)}(r_{12})(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2). \tag{15}$$

One can now make a multipole expansion of this potential and then calculate matrix elements between the appropriate wave functions using the identity

$$\left(\frac{1}{(A!)^{1/2}}\sum_P(-1)^P P\Psi_1(1\cdots A)|\sum_{i<j=1}^A O(i,j)|\frac{1}{(A!)^{1/2}}\sum_P(-1)^P P\Psi_2(1\cdots A)\right)$$

$$\equiv(\Psi_1(1\cdots A)|\sum_{i<j=1}^A O(i,j)|\sum_P(-1)^P P\Psi_2(1\cdots A)). \tag{16}$$

This allows one to separate the interactions into both direct and exchange core-core, core-shell^{-1}, shell^{-1}-shell^{-1}, particle-core, and particle-shell^{-1} terms. The first four of these are easily expressed in terms of reduced matrix elements of the relevant multipole operator and direct and exchange radial matrix elements.[17] For the last term, the interaction of the last particle and the shell^{-1} which has the state $-m_2-\lambda_2$ missing, one finds, due to the

[17] See, for example, J. P. Elliott and A. M. Lane, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. 39, p. 241.

antisymmetry of the wave functions, an expression which contains a sum over all occupied $m_2\lambda_2$ states. If one adds to this sum the term coming from the state $-m_2-\lambda_2$, one has a sum over all $m_2\lambda_2$ and this gives a term diagonal in (l_1j_1), the extra particle state, independent of J and T, and exactly of the same form as the first four terms mentioned above. This additional term must then be subtracted to obtain the correct expression. These remaining terms, which are usually referred to as the "particle-hole interaction" now contain the entire dependence on the coupling, that is on J and T. The off-diagonal matrix elements of H only involve simple matrix elements of v. Since $T(i)$ is a single-particle scalar, it cannot contribute to the off-diagonal matrix elements as long as one of the quantum numbers $l_1j_1l_2j_2$ is changed. Actually, it turns out that in light nuclei, it is sufficient to consider only particle-hole states for which $l_1j_1(l_2j_2)^{-1}$ completely characterize the states. This simplifies things and we will henceforth assume this to be true. The resulting expression, where the diagonal terms have been regrouped and rewritten as the expectation value of H between appropriate wave functions is given exactly by

$$\begin{aligned}(\Phi_{JM_JTM_T}{}^{(n_1'l_1'j_1')(n_2'l_2'j_2')^{-1}}|H|\Phi_{JM_JTM_T}{}^{(n_1l_1j_1)(n_2l_2j_2)^{-1}})&=\delta_{l_2'l_2}\delta_{l_1'l_1}\delta_{j_2'j_2}\delta_{j_1'j_1}\{(\Phi_0|H|\Phi_0)\\&+[(\Phi_{m_1\lambda_1}{}^{n_1l_1j_1}|H|\Phi_{m_1\lambda_1}{}^{n_1l_1j_1})-(\Phi_0|H|\Phi_0)]+[(\Phi_{m_2\lambda_2}{}^{(n_2l_2j_2)^{-1}}|H|\Phi_{m_2\lambda_2}{}^{(n_2l_2j_2)^{-1}})-(\Phi_0|H|\Phi_0)]\}\\&-\sum_{\substack{m_1\lambda_1\\m_2\lambda_2}}\sum_{\substack{m_1'\lambda_1'\\m_2'\lambda_2'}}(-1)^{\lambda_2'-\lambda_2}(-1)^{m_2'-m_2}(-1)^{j_2'-j_2}(j_1'm_1'j_2'm_2'|j_1'j_2'JM_J)(j_1m_1j_2m_2|j_1j_2JM_J)\\&\times(\tfrac{1}{2}\lambda_1'\tfrac{1}{2}\lambda_2'|\tfrac{1}{2}\tfrac{1}{2}TM_T)(\tfrac{1}{2}\lambda_1\tfrac{1}{2}\lambda_2|\tfrac{1}{2}\tfrac{1}{2}TM_T)[(j_2-m_2-\lambda_2j_1'm_1'\lambda_1'|v|j_2'-m_2'-\lambda_2'j_1m_1\lambda_1)\\&\qquad-(j_2-m_2-\lambda_2j_1'm_1'\lambda_1'|v|j_1m_1\lambda_1j_2'-m_2'-\lambda_2')],\end{aligned}\tag{17}$$

where we have introduced the wave function for $A+1$ particles,

$$\Phi_{m_1\lambda_1}{}^{n_1l_1j_1}(1\cdots A+1)=\frac{1}{(A!)^{1/2}}\frac{1}{((A+1)!)^{1/2}}\sum_P(-1)^PP[\Phi_0(1\cdots A)\phi_{m_1\lambda_1}{}^{n_1l_1j_1}(A+1)]\tag{18}$$

in one term above, with a corresponding increase of the number of terms in H. All of the J and T dependence is contained in the last term which involves just simple matrix elements of v, which is to be taken as the free nucleon-nucleon interaction according to our discussion. The extra phases come from counting permutations in taking matrix elements. The expression in brackets above can be identified with the ground-state energy of the nucleus in question with A particles and differences in energy between nuclei with $A\pm1$ particles and the ground state of the A particle nucleus, energy here meaning that energy described by H or the total kinetic and potential energy of the nucleons in the nucleus. This is just the total nuclear mass minus the rest energy of the nucleons making it up. This empirical identification of a large part of the relevant matrix element is the same as that used in the usual intermediate coupling calculation. The term $[(\Phi_{m_1\lambda_1}{}^{n_1l_1j_1}|H|\Phi_{m_1\lambda_1}{}^{n_1l_1j_1})-(\Phi_0|H|\Phi_0)]$ gives the interaction of the last particle with all shells completely closed. The term which must be subtracted to account for this is contained in the matrix elements of v left over. The above expression is evidently independent of m_1 and λ_1. The remaining expression in braces [after canceling the $(\Phi_0|H|\Phi_0)$] is just $(\Phi_{m_2\lambda_2}{}^{(n_2l_2j_2)^{-1}}|H|\Phi_{m_2\lambda_2}{}^{(n_2l_2j_2)^{-1}})$ and gives the interaction between the closed shell and itself, the closed shell and the shell^{-1}, and the shell^{-1} and itself. It is again independent of m_2 and λ_2. We shall call the diagonal term in brackets just $E_0((n_1l_1j_1)(n_2l_2j_2)^{-1})$. It is evidently the unperturbed configuration energy. The angular momentum in the matrix elements of v appearing in the last part of the above expression can be recoupled using standard techniques.[14] We can thus write

$$\begin{aligned}(\Phi_{JM_JTM_T}{}^{(n_1'l_1'j_1')(n_2'l_2'j_2')^{-1}}|H|\Phi_{JM_JTM_T}{}^{(n_1l_1j_1)(n_2l_2j_2)^{-1}})&=\delta_{l_1'l_1}\delta_{l_2'l_2}\delta_{j_1'j_1}\delta_{j_2'j_2}E_0((n_1l_1j_1)(n_2l_2j_2)^{-1})\\&+((n_1'l_1'j_1')(n_2'l_2'j_2')^{-1}JM_JTM_T|v(1,2)|(n_1l_1j_1)(n_2l_2j_2)^{-1}JM_JTM_T),\end{aligned}\tag{19}$$

where

$$\begin{aligned}&((n_1'l_1'j_1')(n_2'l_2'j_2')^{-1}JM_JTM_T|v(1,2)|(n_1l_1j_1)(n_2l_2j_2)^{-1}JM_JTM_T)\\&=-\sum_{\bar J}\sum_{\bar T}(2\bar J+1)(2\bar T+1)\begin{Bmatrix}j_1&j_2&J\\j_1'&j_2'&\bar J\end{Bmatrix}\begin{Bmatrix}\frac{1}{2}&\frac{1}{2}&T\\\frac{1}{2}&\frac{1}{2}&\bar T\end{Bmatrix}\\&\times[((j_1'j_2)\bar J\bar M_J;(\tfrac{1}{2}\tfrac{1}{2})\bar T\bar M_T|v(1,2)|(j_1j_2')\bar J\bar M_J;(\tfrac{1}{2}\tfrac{1}{2})\bar T\bar M_T)-(-1)^{j_1'+j_2-\bar J}(-1)^{\frac{1}{2}+\frac{1}{2}-\bar T}\\&\times((j_2j_1')\bar J\bar M_J;(\tfrac{1}{2}\tfrac{1}{2})\bar T\bar M_T|v(1,2)|(j_1j_2')\bar J\bar M_J;(\tfrac{1}{2}\tfrac{1}{2})\bar T\bar M_T)].\end{aligned}\tag{20}$$

To proceed further we need to calculate the matrix elements of v. In the following we discuss techniques for doing this.

We assume that we can factor out the isospin dependence of $v(1,2)$ as a scalar product of two tensors of rank K in isospace:

$$v(1,2)=v(\mathbf{r}_1,\mathbf{r}_2)W(1,2), \quad \text{where} \quad W=U_K(1)\cdot U_K(2). \tag{21}$$

We define

$$T_K\cdot U_K\equiv\sum_q (-1)^q T_{Kq}U_{K-q}$$

as the scalar product of tensors $T(Kq)$, $U(Kq)$. (Here we only need to consider cases where $K=0$ or 1.)

$$((\tfrac{1}{2}\tfrac{1}{2})\bar{T}\bar{M}_T|W|(\tfrac{1}{2}\tfrac{1}{2})\bar{T}\bar{M}_T)=(-1)^{\frac{1}{2}+\frac{1}{2}-\bar{T}}\begin{Bmatrix}\bar{T} & \frac{1}{2} & \frac{1}{2}\\ K & \frac{1}{2} & \frac{1}{2}\end{Bmatrix}(\tfrac{1}{2}\|U_K(1)\|\tfrac{1}{2})(\tfrac{1}{2}\|U_K(2)\|\tfrac{1}{2}). \tag{22}$$

Doing the sum on $\bar{T}$, we obtain

$$\begin{aligned}&((n_1'l_1'j_1')(n_2'l_2'j_2')^{-1}JM_JTM_T|v(1,2)|(n_1l_1j_1)(n_2l_2j_2)^{-1}JM_JTM_T)\\&=-\sum_{\bar{J}}(2\bar{J}+1)\begin{Bmatrix}j_1 & j_2 & J\\ j_1' & j_2' & \bar{J}\end{Bmatrix}(\tfrac{1}{2}\|U_K(1)\|\tfrac{1}{2})(\tfrac{1}{2}\|U_K(2)\|\tfrac{1}{2})\Bigg[(-1)^{1-T-K}\begin{Bmatrix}\frac{1}{2} & \frac{1}{2} & T\\ \frac{1}{2} & \frac{1}{2} & K\end{Bmatrix}\\&\times((j_1'j_2)\bar{J}\bar{M}|v(\mathbf{r}_1,\mathbf{r}_2)|(j_1j_2')\bar{J}\bar{M})-(-1)^{j_1'+j_2-\bar{J}}\frac{1}{2T+1}\delta_{TK}((j_2j_1')\bar{J}\bar{M}|v(\mathbf{r}_1,\mathbf{r}_2)|(j_1j_2')\bar{J}\bar{M})\Bigg].\end{aligned} \tag{23}$$

Next let us take $v(1,2)$ to be independent of isospin, $W=1$. This is actually no loss of generality since any isospin factor can be reduced to a combination of space- and spin-exchange operators by making explicit use of the antisymmetry of the wave functions. We see that the exchange term vanishes for $T=1$, $K=0$, and the isospin factors give unity in the direct term.

$$\begin{aligned}&((n_1'l_1'j_1')(n_2'l_2'j_2')^{-1}JM_JTM_T|v(1,2)|(n_1l_1j_1)(n_2l_2j_2)^{-1}JM_JTM_T)\\&=-\sum_{\bar{J}}(2\bar{J}+1)\begin{Bmatrix}j_1 & j_2 & J\\ j_1' & j_2' & \bar{J}\end{Bmatrix}((j_1'j_2)\bar{J}\bar{M}|v(\mathbf{r}_1,\mathbf{r}_2)|(j_1j_2')\bar{J}\bar{M}).\end{aligned} \tag{24}$$

One can now proceed in two different ways. First let us suppose $v(\mathbf{r}_1,\mathbf{r}_2)$ is a scalar force with possibly some spin dependence:

$$v(\mathbf{r}_1,\mathbf{r}_2)=v_1(r_{12})+v_2(r_{12})\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2, \quad r_{12}=|\mathbf{r}_1-\mathbf{r}_2|. \tag{25}$$

We evaluate matrix elements of v in terms of Slater integrals as follows. For the direct term we write

$$v_1(r_{12})=\sum_{K=0}^{\infty} f_K(r_1r_2)C_K(1)\cdot C_K(2), \tag{26}$$

where

$$f_K(r_1r_2)=\frac{2K+1}{2}\int_{-1}^{1} v_1(r_{12})P_K(\cos\theta_{12})d\cos\theta_{12}, \quad r_{12}=(r_1^2+r_2^2-2r_1r_2\cos\theta_{12})^{1/2}$$

and

$$C_{Kq}(1)=(4\pi/2K+1)^{1/2}Y_{Kq}(\theta_1,\phi_1).$$

Here $P_K(\cos\theta)$ is the usual Legendre polynomial and the $Y_{lm}(\theta,\phi)$ are spherical harmonics as defined in Ref. 14. Then for the direct term we obtain

$$\begin{aligned}&((n_1'l_1'j_1')(n_2'l_2'j_2')^{-1}JM_JTM_T|v_1(1,2)|(n_1l_1j_1)(n_2l_2j_2)^{-1}JM_JTM_T)\\&=-\sum_{K=0}^{\infty}F_K(-1)^{j_1+j_2+J-K}\begin{Bmatrix}J & j_2' & j_1'\\ K & j_1 & j_2\end{Bmatrix}((l_1'\tfrac{1}{2})j_1'\|C_K(1)\|(l_1\tfrac{1}{2})j_1)((l_2\tfrac{1}{2})j_2\|C_K(2)\|(l_2'\tfrac{1}{2})j_2'),\end{aligned} \tag{27}$$

where

$$F_K=\int_0^\infty\int_0^\infty R_{n_1'l_1'}{}^*(r_1)R_{n_2l_2}{}^*(r_2)f_K(r_1r_2)R_{n_1l_1}(r_1)R_{n_2'l_2'}(r_2)r_1^2dr_1r_2^2dr_2$$

is the usual Slater integral, and $R_{nl}(r)$ denotes the radial wave function. Similarly for the spin-dependent term we can define a tensor operator

$$X_{\lambda\mu}{}^{(K,1)}\equiv\sum_{qq'}C_{Kq}\sigma_{1q'}(Kq1q'|K1\lambda\mu), \tag{28}$$

and therefore

$$v_2(r_{12})\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2=\sum_{K\lambda}(-1)^{K+1-\lambda}f_K(r_1,r_2)X_\lambda^{(K,1)}(1)\cdot X_\lambda^{(K,1)}(2)\,, \tag{29}$$

where again f_K is defined as before in terms of v_2. We obtain

$$((n_1'l_1'j_1')(n_2'l_2'j_2')^{-1}JM_JTM_T|v_2(r_{12})\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2|(n_1l_1j_1)(n_2l_2j_2)^{-1}JM_JTM_T)$$

$$=-\sum_{K\lambda}(-1)^{j_1+j_2+J-K+1}F_K\begin{Bmatrix}J & j_2' & j_1'\\ \lambda & j_1 & j_2\end{Bmatrix}((l_1'\tfrac{1}{2})j_1'\|X_\lambda^{(K,1)}(1)\|(l_1\tfrac{1}{2})j_1)((l_2\tfrac{1}{2})j_2\|X_\lambda^{(K,1)}(2)\|(l_2'\tfrac{1}{2})j_2')\,. \tag{30}$$

Using these results along with the identities

$$\begin{aligned}((l'\tfrac{1}{2})j'\|C_K\|(l\tfrac{1}{2})j)&=(-1)^{j'+\frac{1}{2}}((2j'+1)(2j+1))^{1/2}\begin{pmatrix}j' & K & j\\ \frac{1}{2} & 0 & -\frac{1}{2}\end{pmatrix}\left[\frac{1+(-1)^{l'+l+K}}{2}\right],\\ ((l'\tfrac{1}{2})j'\|X_\lambda^{(K,1)}\|(l\tfrac{1}{2})j)&=((2j'+1)(2j+1)(2\lambda+1))^{1/2}\begin{Bmatrix}l' & l & K\\ \frac{1}{2} & \frac{1}{2} & 1\\ j' & j & \lambda\end{Bmatrix}(l'\|C_K\|l)(\tfrac{1}{2}\|\sigma_1\|\tfrac{1}{2})\,,\end{aligned} \tag{31}$$

one can calculate the matrix elements of v in terms of Slater integrals F_K. For the case of ordinary forces in zero range

$$v(\mathbf{r}_1,\mathbf{r}_2)=v_0\delta(\mathbf{r}_1-\mathbf{r}_2)\,,$$

one obtains

$$F_K=v_0\frac{2K+1}{4\pi}\int_0^\infty R_{n_1'l_1'}{}^*(r)R_{n_2'l_2'}(r)R_{n_1l_1}(r)R_{n_2l_2}{}^*(r)r^2dr\,. \tag{32}$$

An alternative method in the case where we take our single-particle states to be harmonic oscillator wave functions is to reduce the matrix elements to integrals over the relative coordinate (Talmi integrals[18]). This is more useful when $v(\mathbf{r}_1,\mathbf{r}_2)$ involves arbitrary types of exchange forces, and it is particularly convenient for studying the effects of varying the potential well shape $v(|\mathbf{r}_1-\mathbf{r}_2|)$. The procedure is first to go over to LS coupling:

$$\begin{aligned}((j_1'j_2)J\bar{M}|v(\mathbf{r}_1,\mathbf{r}_2)|(j_1j_2')J\bar{M})=\sum_{\substack{LS\\ L'S'}}\,&((2L+1)(2L'+1)(2S+1)(2S'+1))^{1/2}\\ \times((2j_1+1)(2j_1'+1)(2j_2+1)(2j_2'+1))^{1/2}&\begin{Bmatrix}l_1' & l_2 & L\\ \frac{1}{2} & \frac{1}{2} & S\\ j_1' & j_2 & \bar{J}\end{Bmatrix}\begin{Bmatrix}l_1 & l_2' & L'\\ \frac{1}{2} & \frac{1}{2} & S'\\ j_1 & j_2' & \bar{J}\end{Bmatrix}\\ \times((l_1'l_2)L(\tfrac{1}{2}\tfrac{1}{2})S\bar{J}\bar{M}|v(\mathbf{r}_1,\mathbf{r}_2)|(l_1l_2')L'(\tfrac{1}{2}\tfrac{1}{2})S'\bar{J}\bar{M})\,.\end{aligned} \tag{33}$$

For simplicity let us discuss the case where v contains no tensor or spin-orbit forces. Let us assume that we can factor out the spin-dependent part,

$$v(\mathbf{r}_1,\mathbf{r}_2)=V(r_{12})\cdot\sigma(1,2)\,,$$

$$\begin{aligned}&((l_1'l_2)L(\tfrac{1}{2}\tfrac{1}{2})S\bar{J}\bar{M}|v(\mathbf{r}_1,\mathbf{r}_2)|(l_1l_2')L'(\tfrac{1}{2}\tfrac{1}{2})S'\bar{J}\bar{M})\\ &\qquad=\frac{\delta_{LL'}\delta_{SS'}}{((2L+1)(2S+1))^{1/2}}((\tfrac{1}{2}\tfrac{1}{2})S\|\sigma(1,2)\|(\tfrac{1}{2}\tfrac{1}{2})S)((l_1'l_2)L\|V(r_{12})\|(l_1l_2')L)\,.\end{aligned} \tag{34}$$

Next we employ the so-called transformation brackets[19] to separate the center of mass and relative coordinates for two particles in a harmonic oscillator,

$$\begin{aligned}(n_1'n_2(l_1'l_2)L\|V(r_{12})\|n_1n_2'(l_1l_2')L)=\sum_{\substack{N\mathfrak{L}nl\\ N'\mathfrak{L}'n'l'}}\,&(N\mathfrak{L}nl,L|n_1'l_1'n_2l_2,L)\\ \times(N'\mathfrak{L}'n'l',L|n_1l_1n_2'l_2',L)&(Nn(\mathfrak{L}l)L\|V(r_{12})\|N'n'(\mathfrak{L}'l')L)\,.\end{aligned} \tag{35}$$

[18] I. Talmi, Helv. Phys. Acta **25**, 185 (1952).

[19] M. Moshinsky, Nucl. Phys. **13**, 104 (1959). Note that our bracket $(N\mathfrak{L}nl,L|n_1l_1n_2l_2,L)$ differs from Moshinsky's bracket $\langle N\mathfrak{L}nl,L|n_1l_1n_2l_2,L\rangle$ by a phase factor $(N\mathfrak{L}nl,L|n_1l_1n_2l_2,L)=(-1)^{l_2}\langle N\mathfrak{L}nl,L|n_1l_1n_2l_2,L\rangle$ due to the interchange of center-of-mass and relative quantum numbers, $N\mathfrak{L}$ and nl, in the definition.

But

$$(Nn(\mathfrak{L}l)L\|V(r_{12})\|N'n'(\mathfrak{L}'l')L)=\delta_{NN'}\delta_{\mathfrak{L}\mathfrak{L}'}\delta_{ll'}(2L+1)^{1/2}(nl|V(r)|n'l),$$

where

$$(nl|V(r)|n'l)=\int_0^\infty R_{nl}'^*(r)V(r)R_{n'l}'(r)r^2dr \tag{36}$$

and $R_{nl}'(r)$ are harmonic oscillator radial wave functions for a particle of mass $M/2$ (M=nucleon mass);

$$R_{nl}'(r)=\left(\frac{2(n-1)!}{b'^3[\Gamma(n+l+\frac{1}{2})]^3}\right)^{1/2}\left(\frac{r}{b'}\right)^l e^{-\frac{1}{2}(r/b')^2}L_{n-1}^{l+\frac{1}{2}}\left(\frac{r^2}{b'^2}\right), \quad \begin{matrix} n=1,2,3,\cdots \\ l=0,1,2,\cdots, \end{matrix} \tag{37}$$

where $b'=(2\hbar/M\omega)^{1/2}$, $\hbar\omega$=oscillator energy, and $L_\nu^\alpha(z)$ are the Laguerre polynomials. (See Ref. 24.)

Finally we can write

$$(nl|V(r)|n'l)=\sum_p B(nln'l;p)I_p, \tag{38}$$

where

$$I_p=[2/\Gamma(p+\tfrac{3}{2})]\int_0^\infty x^{2p+2}e^{-x^2}V(b'x)dx$$

is defined as the Talmi integral of V. The B coefficients have been computed and tabulated, as well as the transformation brackets.[20] Following the notation of Ref. 20, we define

$$C(n_1'l_1'n_2l_2;n_1l_1n_2'l_2';L,p)=\sum_{N\mathfrak{L}nn'l}(N\mathfrak{L}nl,L|n_1'l_1'n_2l_2,L)(N\mathfrak{L}n'l,L|n_1l_1n_2'l_2',L)B(nln'l;p). \tag{39}$$

We obtain

$$\begin{aligned}&((n_1'l_1'j_1')(n_2'l_2'j_2')^{-1}JM_JTM_T|v(1,2)|(n_1l_1j_1)(n_2l_2j_2)^{-1}JM_JTM_T)\\&=-((2j_1+1)(2j_1'+1)(2j_2+1)(2j_2'+1))^{1/2}\sum_{\bar{J}LS}\begin{Bmatrix}j_1&j_2&J\\j_1'&j_2'&\bar{J}\end{Bmatrix}\begin{Bmatrix}l_1'&l_2&L\\\frac{1}{2}&\frac{1}{2}&S\\j_1'&j_2&\bar{J}\end{Bmatrix}\begin{Bmatrix}l_1&l_2'&L\\\frac{1}{2}&\frac{1}{2}&S\\j_1&j_2'&\bar{J}\end{Bmatrix}\\&\times(2\bar{J}+1)(2L+1)(2S+1)\frac{((\frac{1}{2}\frac{1}{2})S\|\sigma(1,2)\|(\frac{1}{2}\frac{1}{2})S)}{(2S+1)^{1/2}}\sum_p C(n_1'l_1'n_2l_2;n_1l_1n_2'l_2';L,p)I_p(V).\end{aligned} \tag{40}$$

An important feature of this method is that when $v(\mathbf{r}_1,\mathbf{r}_2)$ contains the Majorana space-exchange operator $P_M(1,2)$, we can use the fact that

$$(nl\|V(r_{12})P_M(1,2)\|n'l)=(-1)^l(nl\|V(r_{12})\|n'l)=(-1)^l(2l+1)^{1/2}(nl|V(r)|n'l), \tag{41}$$

since exchanging $\mathbf{r}_1$ and $\mathbf{r}_2$ is equivalent to the parity operator acting on the relative wave function. Therefore the summation in the C coefficient carries an extra factor $(-1)^l$. In particular if $v(\mathbf{r}_1,\mathbf{r}_2)$ is a Serber force, then the terms for the C coefficient only need to be summed over *even* values of l.

The calculation of the diagonal matrix elements E_0 is a straightforward numerical procedure. We define

$$E^{(A)}\equiv W^{(A)}-AMc^2,$$

where $W^{(A)}$ is the total energy of the nucleus (A). Then we have

$$\begin{aligned}(\Phi_{m_1\lambda_1}{}^{(n_1l_1j_1)}|H|\Phi_{m_1\lambda_1}{}^{(n_1l_1j_1)})-(\Phi_0|H|\Phi_0)&=E_{n_1l_1j_1}{}^{(A+1)}-E_0{}^{(A)},\\(\Phi_{m_2\lambda_2}{}^{(n_2l_2j_2)^{-1}}|H|\Phi_{m_2\lambda_2}{}^{(n_2l_2j_2)^{-1}})-(\Phi_0|H|\Phi_0)&=E_{n_2l_2j_2}{}^{(A-1)}-E_0{}^{(A)}.\end{aligned} \tag{42}$$

All of these numbers are chosen directly from the experimental data.[21] The resulting eigenvalues of the secular determinant are the energy differences $[E-(\Phi_0|H|\Phi_0)]$.

Once one has diagonalized the secular matrix and found the coefficients α^K, any observable of the system may

[20] See for example, M. Moshinsky and T. A. Brody, *Tables of Transformation Brackets* (Universidad de Mexico, Mexico, 1960).

[21] T. Lauritsen and F. Ajzenberg-Selove, *Energy Levels of Light Nuclei* (Printing and Publishing Office, National Academy of Sciences—National Research Council, Washington, D. C., 1962).

be calculated. A transition matrix element of a multipole operator $\mathfrak{M}_{JT}$ is given, for example, by

$$|(\Psi_{JT}\|\mathfrak{M}_{JT}\|\Psi_0)|^2=|\sum_{\substack{n_1l_1j_1\\n_2l_2j_2}}\alpha_{JT}{}^{(n_1l_1j_1)(n_2l_2j_2)^{-1}}(n_1l_1j_1\|\mathfrak{M}_{JT}\|n_2l_2j_2)|^2$$
$$=|(\Psi_0\|\mathfrak{M}_{JT}\|\Psi_{JT})|^2, \tag{43}$$

where the symbol $\|$ indicates reduced matrix element with respect to both isotopic spin and angular momentum. Note the matrix elements $(n_1l_1j_1\|\mathfrak{M}_{JT}\|n_2l_2j_2)$ in the sum are just *simple single-particle matrix elements*, and there are no extra phases in this expression.

Let us consider in particular the calculation of the reduced matrix elements of the operator $T_{JM}{}^{\mathrm{el}}(q)$, defined in Sec. II between the ground state and a state of $T=1$, $M_T=0$. This can be written

$$(\mathbf{\Phi}_{J;T=1,M_T=0}{}^{(n_1l_1j_1)(n_2l_2j_2)^{-1}}\|T_J{}^{\mathrm{el}}(q)\|\mathbf{\Phi}_0)=(\tfrac{1}{2})^{1/2}(n_1(l_1\tfrac{1}{2})j_1\|t_{J,v}{}^{\mathrm{el}}(q)\|n_2(l_2\tfrac{1}{2})j_2)$$
$$=(\tfrac{1}{3})^{1/2}(\mathbf{\Phi}_{J;T=1}{}^{(n_1l_1j_1)(n_2l_2j_2)^{-1}}\|T_{J,T=1}{}^{\mathrm{el}}(q)\|\mathbf{\Phi}_0), \tag{44}$$

where $t_{JM,v}{}^{\mathrm{el}}(q)$ is the isovector part of the single-particle operator

$$t_{JM;q}{}^{\mathrm{el}}(\mathbf{r})=\frac{1}{q}\left\{[\nabla\times j_J(qr)\mathcal{Y}_{J1}{}^M(\theta,\phi)]\cdot\frac{\hbar}{Mc}\frac{1+\tau_3}{2i}\nabla+q^2j_J(qr)\mathcal{Y}_{J1}{}^M(\theta,\phi)\cdot\frac{\hbar}{2Mc}\left(\frac{\lambda_p+\lambda_n}{2}+\frac{\lambda_p-\lambda_n}{2}\tau_3\right)\boldsymbol{\sigma}\right\}. \tag{45}$$

Here $\lambda_p=2.8$, $\lambda_n=-1.9$ are the total magnetic moments of the proton and neutron. We define

$$t_{JM}{}^{\mathrm{el}}(q)=\tfrac{1}{2}t_{JM,s}{}^{\mathrm{el}}(q)+\tfrac{1}{2}\tau_3t_{JM,v}{}^{\mathrm{el}}(q). \tag{46}$$

Using standard techniques of tensor algebra (see, for example, Refs. 14 and 23) one can calculate the j and j' dependence of the single-particle matrix elements. Omitting the intermediate steps we obtain finally, for $J=1$;

$$(n'(l'\tfrac{1}{2})j'\|t_{1,v}{}^{\mathrm{el}}(q)\|n(l\tfrac{1}{2})j)=\frac{\hbar}{Mc}\left(\frac{3}{4\pi}(2l+1)(2l'+1)(2j+1)(2j'+1)\right)^{1/2}$$
$$\times\left\{(-1)^{l'+j+\frac{1}{2}}\begin{Bmatrix}l' & j' & \frac{1}{2}\\ j & l & 1\end{Bmatrix}\left[\left(\frac{2}{3}\right)^{1/2}\left[\begin{Bmatrix}0 & 1 & 1\\ l & l' & l+1\end{Bmatrix}\frac{\begin{pmatrix}l' & 0 & l+1\\ 0 & 0 & 0\end{pmatrix}}{\begin{pmatrix}l+1 & 1 & l\\ 0 & 0 & 0\end{pmatrix}}\frac{l+1}{2l+1}\left(n'l+1\left|j_0(qr)\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\times\left(\frac{d}{dr}-\frac{l}{r}\right)\right|nl\right)+\begin{Bmatrix}0 & 1 & 1\\ l & l' & l-1\end{Bmatrix}\frac{\begin{pmatrix}l' & 0 & l-1\\ 0 & 0 & 0\end{pmatrix}}{\begin{pmatrix}l-1 & 1 & l\\ 0 & 0 & 0\end{pmatrix}}\frac{l}{2l+1}\left(n'l-1\left|j_0(qr)\left(\frac{d}{dr}+\frac{l+1}{r}\right)\right|nl\right)\right]\right.$$
$$-\left(\frac{5}{3}\right)^{1/2}\left[\begin{Bmatrix}2 & 1 & 1\\ l & l' & l+1\end{Bmatrix}\frac{\begin{pmatrix}l' & 2 & l+1\\ 0 & 0 & 0\end{pmatrix}}{\begin{pmatrix}l+1 & 1 & l\\ 0 & 0 & 0\end{pmatrix}}\frac{l+1}{2l+1}\left(n'l+1\left|j_2(qr)\left(\frac{d}{dr}-\frac{l}{r}\right)\right|nl\right)\right.$$
$$\left.\left.+\begin{Bmatrix}2 & 1 & 1\\ l & l' & l-1\end{Bmatrix}\frac{\begin{pmatrix}l' & 2 & l-1\\ 0 & 0 & 0\end{pmatrix}}{\begin{pmatrix}l-1 & 1 & l\\ 0 & 0 & 0\end{pmatrix}}\frac{l}{2l+1}\left(n'l-1\left|j_2(qr)\left(\frac{d}{dr}+\frac{l+1}{r}\right)\right|nl\right)\right]\right]$$
$$+\tfrac{1}{2}q(\lambda_p-\lambda_n)(-1)^{l'}(18)^{1/2}\begin{Bmatrix}l' & l & 1\\ \frac{1}{2} & \frac{1}{2} & 1\\ j' & j & 1\end{Bmatrix}\begin{pmatrix}l' & 1 & l\\ 0 & 0 & 0\end{pmatrix}(n'l'|j_1(qr)|nl)\Bigg\}. \tag{47}$$

Here we define, for example,

$$(n'l'|\,j_\lambda(qr)\,|nl)=\int_0^\infty R_{n'l'}{}^*(r)j_\lambda(qr)R_{nl}(r)r^2dr. \tag{48}$$

Note that these radial wave functions are defined the same way as the $R_{nl}'(r)$ of Eq. (37) except that b' is replaced by $b=(\hbar/M\omega)^{1/2}$. These matrix elements when combined with the wave functions allow one to calculate form factors for transverse excitation of the diagonalized 1^-, $T=1$ particle-hole states.

In order to carry out numerical calculations one must choose a set of single-particle wave functions; a convenient and reasonable choice is the harmonic oscillator eigenfunctions. We choose the oscillator parameter by fitting the Coulomb energy differences in mirror nuclei.[22] This procedure gives results similar to the oscillator parameter obtained by fitting to the rms charge radius in light nuclei as observed in elastic electron scattering.[23]

Using these wave functions we can compute the Slater integrals or Talmi integrals directly. In order to calculate the single-particle matrix elements of $t_{1,v}{}^{\rm el}(q)$ we use, in addition, the following formulas satisfied by harmonic oscillator radial wave functions,[24] $R_{nl}(r)$, $n=1, 2, 3, \cdots$:

$$\begin{aligned}
&R_{2l}=(l+\tfrac{3}{2})^{1/2}R_{1l}-(l+\tfrac{5}{2})^{1/2}R_{1l+2},\\
&R_{3l}=((l+\tfrac{5}{2})(l+\tfrac{3}{2}))^{1/2}R_{1l}-2(l+\tfrac{5}{2})R_{1l+2}+((l+\tfrac{9}{2})(l+\tfrac{7}{2}))^{1/2}R_{1l+4},\\
&\left(\frac{d}{dr}-\frac{l}{r}\right)R_{1l}=-\frac{1}{b}(l+\tfrac{3}{2})^{1/2}R_{1l+1},\\
&\left(\frac{d}{dr}+\frac{l+1}{r}\right)R_{1l}=\frac{1}{b}(2(2l+1))^{1/2}R_{1l-1}-\frac{1}{b}(l+\tfrac{3}{2})^{1/2}R_{1l+1}.
\end{aligned} \tag{49}$$

Also

$$\begin{aligned}
(n_2l_2|\,j_\nu(qr)\,|n_1l_1)=&\frac{1}{\Gamma(\nu+\frac{3}{2})}\left(\frac{qb}{2}\right)^{\nu+\frac{1}{2}}\left(\frac{\pi}{2qb}\Gamma(n_1)\Gamma(n_2)\right)^{1/2}(\Gamma(n_1+l_1+\tfrac{1}{2})\Gamma(n_2+l_2+\tfrac{1}{2}))^{1/2}\sum_{m_1=0}^{n_1-1}\sum_{m_2=0}^{n_2-1}\frac{(-1)^{m_1+m_2}}{m_1!m_2!}\\
&\times\frac{1}{\Gamma(n_1-m_1)\Gamma(n_2-m_2)}\frac{\Gamma(\frac{1}{2}(l_1+l_2+2m_1+2m_2+\nu+3))}{\Gamma(m_1+l_1+\frac{3}{2})\Gamma(m_2+l_2+\frac{3}{2})}{}_1F_1(\tfrac{1}{2}(l_1+l_2+2m_1+2m_2+\nu+3);\nu+\tfrac{3}{2};-\tfrac{1}{4}q^2b^2),
\end{aligned} \tag{50}$$

where ${}_1F_1(\alpha;\gamma;z)$ is the confluent hypergeometric function.[25]

Finally we wish to reemphasize the point made earlier concerning the potential $v(r_{12})$; if one takes this model seriously, then v must be identified with the free nucleon-nucleon interaction. One should really not treat this as a parameter, but instead choose it to fit the low-energy n-p scattering data.

IV. COMPARISON WITH EXPERIMENTS

We have carried out the calculations described in the previous section for the case of carbon-12. The oscillator

TABLE I. Energy levels and wave functions for the 1^-, $T=1$ states in C^{12}. The Hamiltonian was diagonalized with a Serber force, Yukawa well residual interaction with parameters chosen to fit low-energy n-p scattering data, as discussed in the text.

	$\Phi((2s_{\frac{1}{2}})(1p_{\frac{1}{2}})^{-1})$ $E_0=16.86$ MeV	$\Phi((1d_{\frac{3}{2}})(1p_{\frac{1}{2}})^{-1})$ $E_0=17.62$ MeV	$\Phi((1d_{\frac{3}{2}})(1p_{\frac{3}{2}})^{-1})$ $E_0=22.11$ MeV	$\Phi((1p_{\frac{3}{2}})(1s_{\frac{1}{2}})^{-1})$ $E_0=30.05$ MeV
$E_1=19.57$ MeV				
Ψ_1:	0.977	−0.168	0.133	−0.016
$E_2=23.26$ MeV				
Ψ_2:	0.194	0.952	−0.211	0.106
$E_3=25.01$ MeV				
Ψ_3:	−0.088	0.252	0.933	−0.243
$E_4=35.80$ MeV				
Ψ_4:	−0.027	−0.044	0.260	0.964

[22] B. C. Carlson and I. Talmi, Phys. Rev. **96**, 436 (1954).
[23] R. Hofstadter, Ann. Rev. Nucl. Sci. **7**, 231 (1957).
[24] R. Willey, Nucl. Phys. **40**, 529 (1963).
[25] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. I, p. 784.

parameter was chosen from Ref. 22 as the average of the parameters for C^{11} and C^{13}; we found this to be $b\cong 1.6$ F. For a set of basis states we chose the four lowest energy shell-model particle-hole states:

$$\Phi_{J=1^-,T=1}{}^{(2s_{\frac{1}{2}})(1p_{\frac{3}{2}})^{-1}};\quad E_0((2s_{\frac{1}{2}})(1p_{\frac{3}{2}})^{-1})=18.72-1.86=16.86\text{ MeV}.$$

$$\Phi_{J=1^-,T=1}{}^{(1d_{\frac{5}{2}})(1p_{\frac{3}{2}})^{-1}};\quad E_0((1d_{\frac{5}{2}})(1p_{\frac{3}{2}})^{-1})=18.72-1.10=17.62\text{ MeV}.$$

$$\Phi_{J=1^-,T=1}{}^{(1d_{\frac{3}{2}})(1p_{\frac{3}{2}})^{-1}};\quad E_0((1d_{\frac{3}{2}})(1p_{\frac{3}{2}})^{-1})=18.72+3.39=22.11\text{ MeV}.$$

$$\Phi_{J=1^-,T=1}{}^{(1p_{\frac{1}{2}})(1s_{\frac{1}{2}})^{-1}};\quad E_0((1p_{\frac{1}{2}})(1s_{\frac{1}{2}})^{-1})=35-4.95=30.05\text{ MeV}.$$

For a potential $v(1,2)$ we chose a Serber force with parameters adjusted to fit low-energy n-p scattering data;

$$v(1,2)=[{}^1v(r_{12}){}^1P+{}^3v(r_{12}){}^3P](\tfrac{1}{2}[1+P_M(1,2)]),\quad (51)$$

where

$${}^1P=\tfrac{1}{4}(1-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2),\quad {}^3P=\tfrac{1}{4}(3+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)$$

and $P_M(1,2)=$Majorana exchange operator. As a first choice we took $v(r_{12})$ to be a Yukawa well; the parameters for this case are given in Table VI of Ref. 15.

$$v(r_{12})=V_0\frac{e^{-\mu r_{12}}}{\mu r_{12}}\quad {}^1V_0=-46.87\text{ MeV},\quad {}^1\mu=0.8547\text{ F}^{-1},$$

$${}^3V_0=-52.13\text{ MeV},\quad {}^3\mu=0.7261\text{ F}^{-1}.$$

With this choice of potential we computed the Talmi integrals I_p and the matrix elements of v between the basis states. The 4×4 Hamiltonian matrix was then constructed and diagonalized with the aid of the IBM-7090 computer at the Stanford University Computation Center.

The resulting eigenvalues and eigenstates are given in Table I.

Experimentally the giant resonance is observed as a single broad peak with a maximum at 22.5 MeV in the photoabsorption and (p,γ_0) experiments.[26] In the electron scattering experiments[27] a broad peak is also observed in the same energy region, but with its maximum shifted upward to about 24.5 MeV. Therefore we interpret the dipole transitions in this energy region in terms of the two states Ψ_2 and Ψ_3. Note that we have assumed that practically all of the transition strength observed in electron scattering in the giant resonance region comes from $E1$ transitions. The single-particle Weisskopf estimates for $M1$, $E2$, and $M2$ transitions in electron scattering at $q=120$ MeV/c and $E_{fi}=20$ MeV are 10 to 15% of the experimental cross sections observed in electron scattering data at these momentum transfers for the giant resonance.[12,27]

[26] H. E. Gove, A. E. Litherland, and R. Batchelor, Nucl. Phys. **26**, 480 (1961).
[27] J. Goldemberg (to be published).

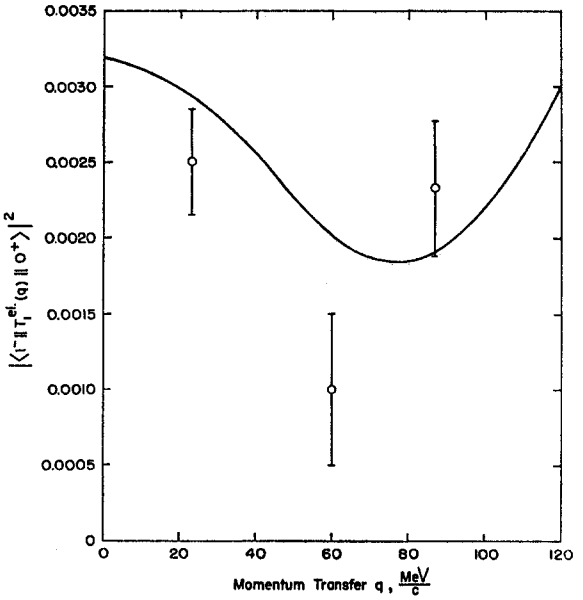

FIG. 1. Squared form factor versus momentum transfer for the giant dipole resonance in carbon-12.

Finally we calculate the squared inelastic form factors for transverse electric excitations of each of these states using the techniques of Sec. III. The total cross section for transitions into the giant resonance is then proportional to the sum of the absolute squares of the form factors $|(J=1^-,T=1\|T_1^{el}(q)\|J=0^+,T=0)|^2$ for the states Ψ_2 and Ψ_3. We call this sum the "square of the form factor of the giant resonance"; this quantity is plotted as a function of q in Fig. 1.

As was pointed out in Sec. II, the cross section for inelastic electron scattering at 180° depends only on the transverse multipole excitations. For excitation of the giant resonance $(J=1^-)$ only $T_1^{el}(q)$ contributes and the cross sections can be used to calculate the matrix elements of this quantity directly. These cross sections have been measured and reported in Refs. 11 and 12 for incident electron energies of 41.5 and 55 MeV. The two resulting squared form factors for momentum transfers 60 and 87 MeV/c are also shown in Fig. 1. (We have chosen a "mean excitation energy" of 23 MeV in our kinematical calculations.)

Finally we have used the formula (4) given in Sec. II to obtain the experimental value of the squared form factor at a momentum transfer of 23 MeV/c. The integrated photoabsorption cross section over the giant resonance in carbon has been measured by many people both for (γ,n) and (γ,p) [or, (p,γ_0)] reactions. We have taken an average of several reported measurements of the (γ,n) cross section[28–30] and also of the (γ,p) [or

[28] R. Montalbetti, L. Katz, and J. Goldemberg, Phys. Rev. **91**, 659 (1953).
[29] R. Nathans and J. Halpern, Phys. Rev. **93**, 437 (1954).
[30] W. C. Barber, W. D. George, and D. D. Reagan, Phys. Rev. **98**, 73 (1955).

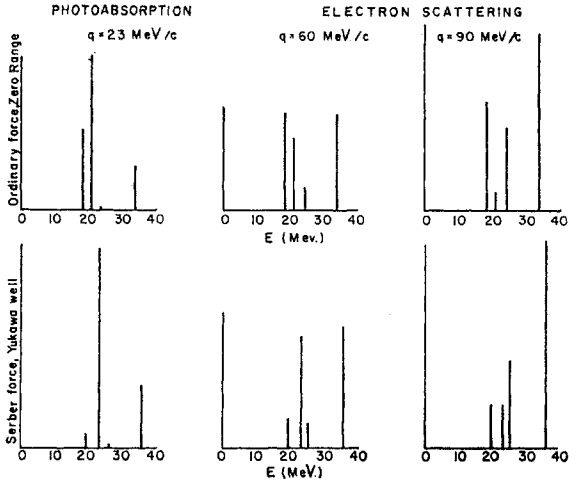

FIG. 2. Relative transition probabilities for transverse excitation of the 1^-, $T=1$ states in carbon-12. The length of each line is proportional to the square of the form factor for excitation of the corresponding state at the given energy and momentum transfer. The two types of residual interaction indicated are discussed in the text.

(p,γ_0)] cross section[26,28,31,32] and added the results. [The (p,γ_0) data were related to the (γ,p) cross section by means of the detailed balance theorem.] The resulting squared form factor is shown in Fig. 1. The error bars on this point are drawn to represent the degree of variation in the results obtained by various different authors.

We wish to point out that the theory predicts that there is a dip in the squared form factor which is clearly seen experimentally. This arises because the giant resonance is composed of two states, Ψ_2 and Ψ_3. As the momentum transfer q increases the squared form factor of Ψ_3 increases while that of Ψ_2 decreases. The sum therefore goes through a minimum.

This can also be clearly seen in Fig. 2, where we have indicated the squared form factors, or relative transition strengths, for all four states at several values of q. (These are labeled "Serber Force, Yukawa Well.") The theory predicts that there is a peak in the cross section as a function of energy, and also that this peak shifts upward by about 2 MeV as q increases from 23 MeV/c to $\sim$90 MeV/c. A comparison of the photoabsorption data[26,28–32] with the electron scattering data of Ref. 12 confirms the existence of this shift in the position of the resonance. Again, this shift arises because as q increases there is a shift of transition strength from Ψ_3 to Ψ_2.

The observed $(1^-, T=1)$ state at 17.2 MeV in[21] C^{12} has also been resolved in the (p,γ_0) experiments[26] as well as in some of the electron scattering experiments.[12] We describe this state by the wave function Ψ_1, and we have plotted the squared form factor for excitation of this state in Fig. 3 (labeled "Serber force, Yukawa well").

By starting with the relative $B^{11}(p,\gamma_0)C^{12}$ cross sections in which the intermediate state of carbon-12 is the giant resonance or the 17.2-MeV state (see, for example, the γ-ray deexcitation curves shown in Ajzenberg-Selove, Ref. 21) and using the detailed balance theorem, we have estimated the integrated photoexcitation cross section for the 17.2-MeV state to be about 5% of that for the giant resonance. The resulting squared form factor for the 17.2-MeV state is shown in Fig. 3. Finally we have used the electron scattering data in Ref. 12 to estimate the cross section for excitation of the 17.2-MeV state at 90-MeV/c momentum transfer, and the resulting squared form factor is also shown in Fig. 3. We see that the tendency for the squared form factor to increase with q as predicted by the theory is reproduced by the experiments.

Finally let us point out that the theory predicts a sizeable form factor for the state at $\sim$36 MeV which we describe as Ψ_4, and in fact this form factor should grow very large at larger momentum transfers (see Fig. 2). Some evidence for the existence of this state has been reported from (p,γ_0) experiments,[33] and preliminary electron scattering results at[27] $q\approx 100$ MeV/c definitely indicate a large bump at about the right energy to correspond to this state. Let us emphasize here that the predictions of the theory for all four states

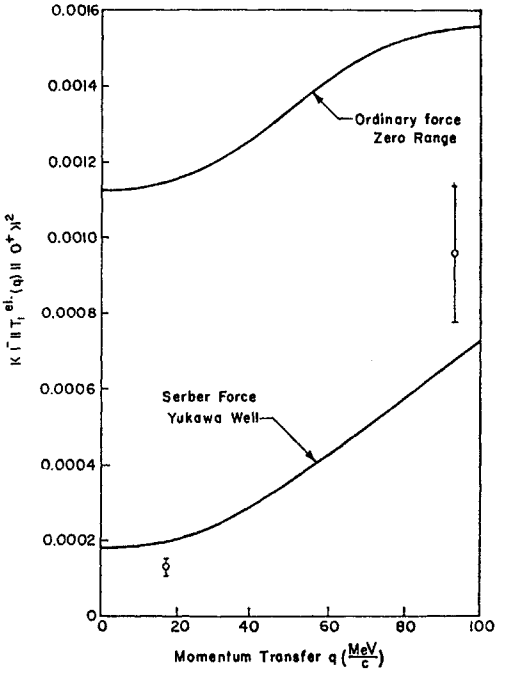

FIG. 3. Squared from factor versus momentum transfer for the $J=1^-$, $T=1$ state at 17.2 MeV in carbon-12. The two types of residual interaction indicated are discussed in the text.

[31] V. J. Vanhuyse and W. C. Barber, Nucl. Phys. 26, 233 (1961).
[32] W. R. Dodge, Stanford University, W. W. Hansen Laboratories of Physics, H.E.P.L. Report No. 246, 1961 (unpublished).
[33] N. W. Reay, N. M. Hintz, and L. L. Lee, Jr., Nucl. Phys. 44, 338 (1963).

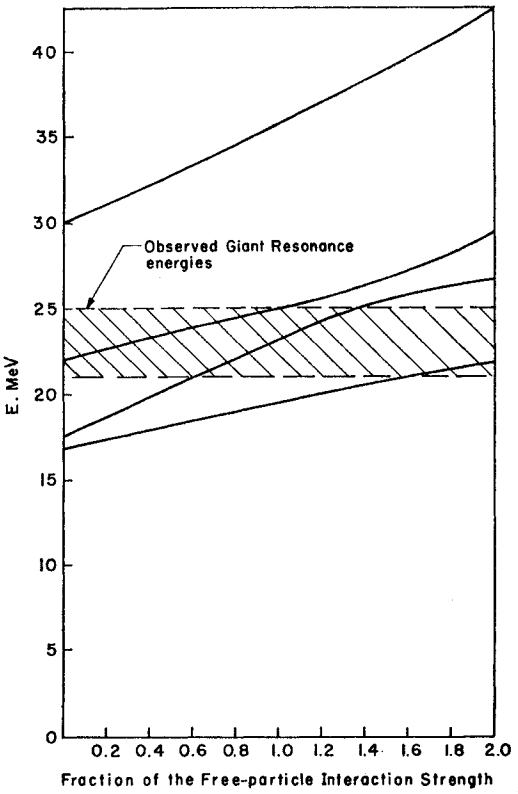

FIG. 4. Energy levels of the 1^-, $T=1$ states in carbon-12 as a function of the residual interaction strength. The interaction used was a Serber force with a Yukawa potential well. We have also indicated the range of energies occupied by the giant resonance in C^{12} as observed in electron scattering.

agree with the experiments both in energies as well as in the form factors. At this point we have a theory with *no* adjustable parameters; the unperturbed configuration energies come from the energies of C^{11} and C^{13}, the oscillator parameter (the only nuclear parameter involved) comes from fitting Coulomb energies, and the two-particle force is taken from free nucleon-nucleon scattering.

We now want to discuss the sensitivity of this result to the details of the theory. In particular we are interested in the effects of variations in the form of $v(1,2)$, both in order to study the sensitivity to different types of nucleon-nucleon potentials and to compare with the calculations made by other authors.

The calculations discussed so far have all been made using the Yukawa potential well shape described earlier in the two-particle interaction $v(1,2)$. We have also calculated the Talmi integrals using a square well with parameters again taken from Table VI, Ref. 15, to fit low-energy n-p scattering data. The resulting energy eigenvalues and eigenvectors are the same within a few percent as those using a Yukawa well.

Furthermore, we have studied the dependence of the energy eigenvalues on the interaction strength by replacing $v(1,2)$ by $\lambda v(1,2)$ and varying the matrix elements by varying λ and holding the configuration energies fixed. The resulting energy levels are shown as functions of λ in Fig. 4.

We have repeated the entire calculation using other types of forces for the form of $v(1,2)$, not necessarily chosen to fit the free nucleon-nucleon interaction. One choice we studied was the "ordinary force with zero range,"

$$v(1,2)=-v_0\delta(\mathbf{r}_1-\mathbf{r}_2)$$

similar to the calculations described in Ref. 7. Here again we calculated the matrix elements of $v(1,2)$ between the same basis states, this time using the technique of Slater integrals described in Sec. III along with the same oscillator wave functions. These were expressed in terms of v_0 which was left as a free parameter. The resulting 4×4 matrix was diagonalized as before for various values of v_0.

The energy levels are plotted in Fig. 5 as functions of $\lambda\equiv v_0/b^3\times10^{-3}$, where b is the oscillator parameter discussed before. We have also indicated in this figure those values of λ which one would compute from low-energy n-p scattering data by averaging $v_0=-\int v(\mathbf{r})d^3r$ over the various potentials given in Table VI, Ref. 15. These give us the singlet parameter $\bar{\lambda}_s$ and the triplet parameter $\bar{\lambda}_t$; we define $\bar{\lambda}=\frac{1}{4}\bar{\lambda}_s+\frac{3}{4}\bar{\lambda}_t$. All three values of λ are shown in Fig. 5.

The parameter λ was then chosen to fit the known $T=1$, $J=1^-$ energy levels in carbon-12. We took $\lambda=0.10$ MeV, or

$$v_0=410\text{ MeV F}^3,$$

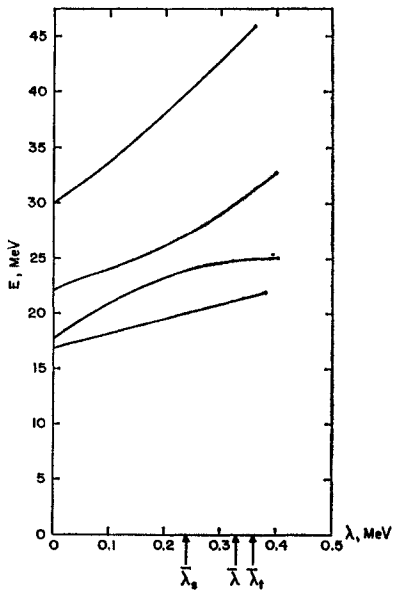

FIG. 5. Energy levels of the 1^-, $T=1$ states in carbon-12 as a function of the residual interaction strength. The interaction used was an ordinary force with zero range. The parameter λ is discussed in the text. We have also shown the values of λ corresponding to the singlet and triplet parts of the free nucleon-nucleon interaction.

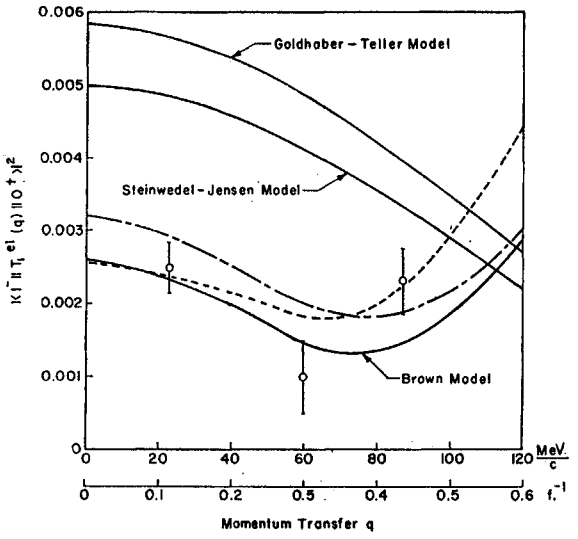

FIG. 6. Squared form factor versus momentum transfer for the giant dipole resonance in carbon-12. The solid curve is calculated using a spin-dependent force with zero range for the residual interaction. The dashed curve is calculated using an ordinary force with zero range for the residual interaction and leaving the high-lying basis state at 30 MeV out of the calculation completely. The long dash-short dash curve is calculated using a Serber force with a Yukawa potential well for the residual interaction; this curve is the same as in Fig. 1. We have also shown the curves calculated using the collective models.

which gave us two levels, as shown, at 24.01 and 20.84 MeV in the observed region of energy of the giant resonance and also two levels at 18.24 and 33.76 MeV. (This value of v_0 is about the same as that chosen in Ref. 7.) The squared inelastic form factors for all four states were then calculated, and are shown in Fig. 2. The squared form factor of the 18.24-MeV state which we identify with the observed state at 17.2 MeV is also shown in Fig. 3, and the squared form factor is a factor of 4–6 too large in magnitude with the δ-function force. This result indicates that the structure of the 17.2-MeV state is very sensitive to the two-body force used. The squared form factor of the giant resonance is again taken to be the sum of the squared form factors of the two states in the giant resonance region, and the result is the same to within a few percent as the curve for the spin-dependent zero range force shown in Fig. 6. Comparison with the result for the Serber force indicates that the form factor for the giant resonance is not particularly sensitive to the two-body force used. We wish to emphasize that the method of choosing v_0 as described above yields a value of λ which is at least 2 or 3 times smaller than the values $(\bar{\lambda},\bar{\lambda}_s,\bar{\lambda}_t)$ corresponding to the free nucleon-nucleon interaction.

This calculation was also modified in several ways. First we left the high-lying basis state $\Phi_{J=1^-,T=1}{}^{(1p\frac{1}{2})(1s\frac{1}{2})^{-1}}$ out of the calculation completely and diagonalized the interaction among the three remaining basis states. Choosing the same value of v_0 as before we obtain three states at 24.49, 20.88, and 18.25 MeV. Choosing the cross section of the giant resonance again to be the sum of the cross sections to the first two states we obtained a squared form factor which is shown in Fig. 6. This gives an indication of the sensitivity of the calculation to higher basis configurations.

Secondly, we chose $v(1,2)$ to be a spin-dependent force with zero range,

$$v(1,2)=-v_0[(1-\eta)+\eta\sigma_1\cdot\sigma_2]\delta(\mathbf{r}_1-\mathbf{r}_2)$$

with η chosen so as to yield the same singlet/triplet ratio for the volume integral of the potential as that of the free interaction. Using parameters chosen from Ref. 34 we obtain a value $\eta=0.064$. This potential was then diagonalized among the four basis states. The resulting eigenvalues and eigenstates were identical to within a few percent to those from the ordinary force zero-range calculation. The form factor of the giant resonance for the spin-dependent force calculation is also shown in Fig. 6. This calculation was also reported in Ref. 12.

Finally we chose $v(1,2)$ to be an "ordinary force, finite range"; in particular, we chose a Gaussian well

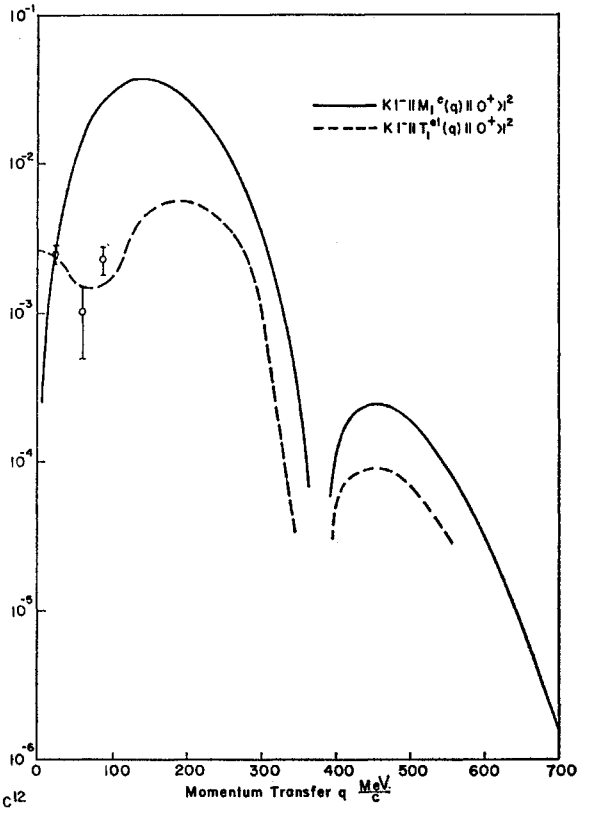

FIG. 7. Longitudinal and transverse squared form factors versus momentum transfer for the giant dipole resonance in carbon-12. The curves are calculated using a spin-dependent force with zero range for the residual interaction.

[34] L. Hulthén and M. Sugawara, *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. 39, p. 1.

shape

$$v(1,2) = -V_0 e^{-(r/a)^2}.$$

Here we chose $a=1.43$ F to fit low-energy scattering data for the range of the triplet part of the n-p force. Matrix elements were computed using the Slater integral techniques discussed in Sec. III and the Hamiltonian was diagonalized among the four basis states. By choosing $V_0=20$ MeV we obtained states at 19.22, 20.84, 24.55, and 34.34 MeV. The volume integral of this potential is 80% of that chosen in the zero-range calculations. Note that if one analyzes low-energy n-p scattering data with a Gaussian potential, one obtains $V_0=79$ MeV. (See Refs. 34, 35.) The wave functions obtained in this calculation resemble those of the Serber force calculation rather than those of the zero-range force calculations, and the squared form factors are therefore also within a few percent of the Serber-force squared form factors.

One can therefore conclude that the type of exchange mixture in the force influences primarily the position of the energy levels as a function of interaction strength, while the wave functions themselves are more sensitive to the range of the force.

The form factors discussed up to now have all been for transverse excitation of the levels. We have also calculated the form factor for Coulomb excitation of the giant resonance using the states obtained from the spin-dependent force calculation; these form factors are the reduced matrix elements of the operator $M_1(q)$, where $M_J(q)$ was defined in Sec. II. The transverse and longitudinal squared form factors for the giant resonance in C^{12} are shown in Fig. 7 out to large values of q.

Note that both squared form factors show the appearance of a minimum in the region $q=200$–300 MeV/c. This minimum occurs because the formula (50) for the radial matrix elements of $j_\nu(qr)$ generally contains zeros from the hypergeometric functions, and the zeros for the various different terms tend to be near each other.

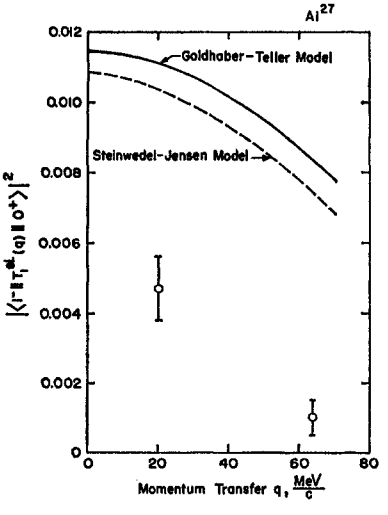

FIG. 8. Squared form factor versus momentum transfer for the giant dipole resonance in aluminum-27. The solid curve is calculated using the Goldhaber-Teller model and the dashed curve is calculated using the Steinwedel-Jensen model.

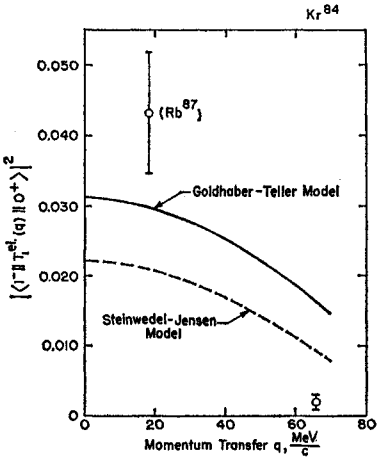

FIG. 9. Squared form factor versus momentum transfer for the giant dipole resonance in krypton-84. The solid curve is calculated using the Goldhaber-Teller model and the dashed curve is calculated using the Steinwedel-Jensen model. The experimental point at $q=18$ MeV/c is the squared form factor measured in rubidium-87.

We have also calculated the form factor of the 17.2-MeV state for both transverse and Coulomb excitation at large values of q. One finds at $q\approx 500$–700 MeV/c that the squared form factors of this state are several times larger than those of the giant resonance, and also that the longitudinal form factor is dominant over the transverse form factor. Preliminary inelastic electron scattering data in carbon-12 using incident electron energies of 800 MeV have been taken by Crannell at the Stanford Mark III linear accelerator. At $\sim 40°$ scattered electron angle (i.e., momentum transfers $q\simeq 500$–600 MeV/c) the data show a peak in the cross section at a scattered electron energy 18 MeV down from the elastic peak. An analysis of this data was made by neglecting the transverse excitation and using the formula in Sec. II to obtain the longitudinal squared form factor from the scattering cross section. The results at present are in agreement with the theory within experimental error.

Finally we have also indicated in Fig. 6 the squared form factors of the giant resonance obtained from collective motion models. The Goldhaber-Teller model (Ref. 1) calculation is discussed in Ref. 11. We have also shown the squared form factor obtained from the Steinwedel-Jensen model[36] calculation discussed in Appendix A. Note that this theory contains one parameter, the symmetry energy constant K from the semi-empirical mass formula. In order to obtain the giant

[35] J. M. Blatt and J. D. Jackson, Phys. Rev. **76**, 18 (1949).

[36] H. Steinwedel and J. H. D. Jensen, Z. Naturforsch. **5a**, 413 (1950).

resonance at an energy of 23 MeV it was necessary to choose $K=15$ MeV, which is 60% of the value obtained in other experiments.[37]

The collective model calculations have also been compared with the electron scattering data for Al^{27} and Kr^{84} which were reported in Ref. 11. The squared form factors are shown in Figs. 8 and 9. The symmetry energy parameter for these two nuclei yielded the correct energy for the same value of K as given in Ref. 37. The data points for the lower values of q were taken from photoabsorption data given in Ref. 28. These authors did not report any results for Kr^{84}, but instead we have plotted the point corresponding to Rb^{87}. This point appears to give an anomalously large value for the squared form factor, and one of the authors in Ref. 28 (J. Goldemberg) has informed us that these experiments are being repeated and photoabsorption data are being taken for Kr^{84}.

V. DISCUSSION AND SUMMARY

It is perhaps a good idea to first summarize the results of the present paper. The Brown theory has been reformulated in terms of the usual Tamm-Dancoff shell-model calculation. The two-particle potential in the nuclear Hamiltonian is taken to be that obtained in free nucleon-nucleon scattering. The calculations of Brown *et al.*[7] are repeated with this two-nucleon force for C^{12}. The ground state of C^{12} is taken to be $(1s_{\frac{1}{2}})^4(1p_{\frac{3}{2}})^8$. The $T=1$, $J^\pi=1^-$ states are taken as linear combinations of the lowest lying states of $T=1$, $J^\pi=1^-$ in C^{12} which are $(2s_{\frac{1}{2}})(1p_{\frac{3}{2}})^{-1}$, $(1d_{\frac{5}{2}})(1p_{\frac{3}{2}})^{-1}$, $(1d_{\frac{3}{2}})(1p_{\frac{3}{2}})^{-1}$, and $(1p_{\frac{1}{2}})(1s_{\frac{1}{2}})^{-1}$. The Hamiltonian matrix between these states is then computed, and most of the interaction matrix elements can be identified with energy differences of C^{12} from C^{11} and C^{13}. This identification comes from merely rewriting many of the diagonal terms of the matrix. A similar identification of such terms was also used in the intermediate coupling calculations of Elliott and Flowers. The off-diagonal matrix elements, for the class of states considered for the $T=1$, $J^\pi=1^-$ states in light nuclei, only involve simple matrix elements of the two-particle potential. The radial wave functions are taken to be harmonic oscillator wave functions in computing the particle-hole matrix elements and the oscillator parameter is taken from fitting Coulomb energies. The resulting calculation of energy levels by diagonalization of the matrix and form factors from the wave functions obtained contains no free parameters. All the input data is taken from other experiments. The resulting four dipole states are at energies 19.57, 23.26, 25.01, and 35.80 MeV. The first state is identified as the known $T=1$, $J=1^-$ state at 17.2 MeV in C^{12}. It is thus 2.4 MeV too high. The computed transverse electric form factor of this state turns out to be in very good agreement with the experimental results of Goldemberg *et al.* and the one point on the form factor curve which can be obtained from photoabsorption. The sum of the two squared transverse electric dipole form factors for the 23.26 and 25.01 states is also in agreement with the photoabsorption and electron scattering data. Moreover, it shows a quite remarkable dip in the region $q<150$ MeV/c which is observed experimentally. The analysis of the contributions of the two states (Fig. 2) shows that in photoabsorption, one should just see one large peak at 23.26 MeV (this is actually observed and comes at 22.55 MeV) with only perhaps a very small shoulder at 25.01 MeV. As the momentum transfer is changed, however, the strength of the 23-MeV level *decreases* while that of the 25 MeV increases leading not only to the dip described above, but also to an *upward shift of 2 MeV in the resonance energy* which is also observed experimentally. The state at 35.80 MeV has been observed in (p,γ) work. Our calculations indicate, however, that for $q>150$ MeV/c this state actually has most of the dipole strength. There are preliminary indications that this is observed in the electron scattering. Attempts to fit the giant resonance form factor in C^{12} with either the Goldhaber-Teller or Steinwedel-Jensen collective models lead to completely wrong form factors. The two main points we want to make in the paper are:

(1) The electron scattering form factors are very sensitive to the structure of the states involved indicating again that electron scattering is a tremendously powerful tool in studying nuclear structure.

(2) The particle-hole picture of the giant resonance seems to be in quite remarkable agreement with the main features of inelastic $T=1$, $J^\pi=1^-$ form factors.

To determine the sensitivity of our result to the force used we carried out several calculations with different forces. These calculations are described in Sec. IV. The main results can be summarized as follows:

(1) The form factor for the giant resonance was quite insensitive to the force used. The dip appeared in every calculation we made. The energy of the state was sensitive to the force. A delta-function interaction had to have a strength much less than a comparative volume integral of the free nucleon-nucleon force to get the levels in the right place, while with the free two-nucleon Serber force (which has no interaction in the odd angular momentum states) the levels came out correctly. Adding the spin dependence of the free nucleon-nucleon force changed almost nothing.

(2) The form factor of the 17.2-MeV state turned out to be very sensitive to the interaction since this state gets most of its strength from the admixture of the higher lying d-particle states. Again the free Serber force gave a form factor in agreement with experiment while the δ-function force gave a result 4 to 6 times too large.

To determine the sensitivity of our result to the type

[37] A. E. S. Green, Phys. Rev. **95**, 1006 (1954).

of states admixed we carried out one calculation of the giant resonance state leaving out the $(1p_{\frac{1}{2}})(1s_{\frac{1}{2}})^{-1}$ state with $E_0=30.05$ MeV and the result is shown in Fig. 6. The difference is appreciable at large q and indicates that for any more really quantitative fit to the data one must include two-particle two-hole states etc., although work by Brown shows such states do not play a dominant role in $T=1$ states.[7] There is one other piece of evidence that such states may be important in the ground state. If one tries to fit the inelastic magnetic dipole form factor of the $T=1$, $J=1^+$ state at 15.1 MeV in C^{12}, which according to Brown[7] is a *pure* $(1p_{\frac{1}{2}})(1p_{\frac{3}{2}})^{-1}$ state, then one obtains a squared form factor which is about a factor of 4 too large. The intermediate coupling calculation of Kurath, which contains a large admixture of other types of states however, gives the correct γ-absorption cross section for this 15.1-MeV state. One is tempted to conclude that the Brown theory is just much more successful for the 1^- states because these states must be made up of particles promoted to the next oscillator shell and can't be obtained by merely recoupling particles within the p shell. It should be mentioned here that calculations of $T=1$, 1^- levels in Pb^{208} have been carried out[38] using the particle-hole theory, and good agreement with experiment is obtained without the inclusion of ground-state correlations. However, calculations of other states $(3^-,4^+)$ in Pb^{208} using the particle-hole theory[39,40] yield considerably poorer agreement with experiment, which supports the conclusion that the restriction to one-particle, one-hole states is particularly successful only for $T=1$, 1^- states. It has also been shown[41] that the inclusion of two-particle, two-hole states can be significant for other nuclei with unfilled shells.

In our calculations we have not treated the center-of-mass motion correctly. Elliott and Flowers did this, and found the results were only important in $T=0$ states. This is evident since the spurious 1^- state corresponding to pure center-of-mass motion must be $T=0$ since any other T cannot be just a translation of the whole system.

Also let us emphasize that the form factors which we have computed are for transverse excitations, and are therefore dependent on the details of the nuclear convection current and magnetization density [Eq. (3)]. It is the matrix elements of these operators which one measures experimentally in photoexcitation and 180° electron scattering, and which are successfully predicted by the Brown theory. However, for small values of $q_{fi}=E_{fi}/\hbar c$ one can apply the equation of current conservation in order to rewrite the transverse electric dipole moment operator for photoabsorption (Siegert's theorem):

$$T_{1M}^{e1}(q_{fi})\cong-(\sqrt{2}/3)q_{fi}\int d^3\mathbf{x}\times\rho_N(\mathbf{x})Y_{1M}(\Omega_{\mathbf{x}}), \quad (52)$$

which only depends on the nuclear charge density. If one now uses the Brown theory to calculate matrix elements of this quantity for photoexcitations into the giant resonance in order to compare with the experimental data, one finds that the Brown theory gives a squared form factor which is larger than experiment by a factor of ~ 2. One could conclude from this that the Brown theory correctly predicts the matrix elements of the current and magnetization, but it fails to give the right matrix elements of the transition charge density. Another way of looking at this is to compare the expression for $T_{1M}^{e1}(q_{fi})$ [Eq. (52)] with the matrix element predicted by the Brown theory:

$$(1^-\|T_1^{e1}(q_{fi})\|0^+)\cong-\frac{\sqrt{2}}{3}\left(\frac{\omega_{\rm osc}}{c}\right)\times\int d^3\mathbf{x}(1^-\|x\rho_N(\mathbf{x})Y_1(\Omega_{\mathbf{x}})\|0^+). \quad (53)$$

Here $\hbar\omega_{\rm osc}$ is the energy parameter used in defining the harmonic oscillator single-particle wave functions. We see that the Brown theory replaces $\hbar cq_{fi}=E_{fi}$ by $\hbar\omega_{\rm osc}$, so that on squaring one obtains a factor

$$\left(\frac{\hbar\omega_{\rm osc}}{E_{fi}}\right)^2=\left(\frac{15\text{ MeV}}{23\text{ MeV}}\right)^2\approx 0.4.$$

This is essentially the origin of the factor of 2 discrepancy mentioned above. If one had really constructed exact eigenstates of the total Hamiltonian, the two methods of calculation [Eq. (52)] must of course give the same results. This is a drawback with any theory that uses approximate wave functions. We should emphasize that we must deal with the transverse multipoles of the current and magnetization if we want to compare with electron scattering since there one cannot work in the long-wavelength limit. We should also add that the current operator which we used had no specific meson-exchange current contributions, but only contained convection and magnetization parts. The question of the contribution of meson-exchange currents to the transverse form factors is still an open one.

One of the most important successes of the Brown theory, as was mentioned before, is the prediction of the dip observed in the squared transverse electric form factor of the giant resonance, which occurred in every shell-model calculation which we made. This basic feature of the shell model can be understood qualitatively in the following way. Let us, for simplicity, turn off the two-particle residual interaction so that

[38] V. V. Balashov, V. G. Shevchenko, and N. P. Yudin, Zh. Eksperim. i Teor. Fiz. **41**, 1929 (1961) [translation: Soviet Phys.—JETP **14**, 1371 (1962)].
[39] J. C. Carter, W. T. Pinkston, and W. M. True, Phys. Rev. **120**, 504 (1960).
[40] W. T. Pinkston, Nucl. Phys. **37**, 312 (1962).
[41] M. V. Mikhailovic and M. Rosina, Nucl. Phys. **40**, 252 (1963).

Ψ_2 and Ψ_3 are simply the basis states $\Phi_{J=1^-,T=1}{}^{(1d_{\frac{5}{2}})(1p_{\frac{1}{2}})^{-1}}$ and $\Phi_{J=1^-,T=1}{}^{(1d_{\frac{3}{2}})(1p_{\frac{1}{2}})^{-1}}$, respectively. (These basis states are, in fact, the principal components of Ψ_2 and Ψ_3 as seen in Table I, for example.) The form factors of these states can be separated into two contributions, one coming from the convection current $\mathbf{j}_N$, and the other arising from the intrinsic magnetization density $\mathbf{\mu}_N$ [see Eq. (3)]. For small values of momentum transfer (i.e., $0\lesssim q\lesssim 100$ MeV/c) the absolute magnitudes of these two contributions behave differently from each other (they behave roughly the same way for both states). The convection current contribution decreases somewhat in absolute magnitude as q increases, while the magnetization contribution starts at zero and increases rapidly in absolute magnitude as q increases. For the state $\Phi_{J=1^-,T=1}{}^{(1d_{\frac{5}{2}})(1p_{\frac{1}{2}})^{-1}}$ these two contributions add *destructively*, so that the squared form factor of this state is due to a large convection current contribution at small q, and rapidly decreases as q increases, actually reaching a minimum at $q\sim 100$ MeV/c (for larger q the squared form factor increases again). The other state $\Phi_{J=1^-,T=1}{}^{(1d_{\frac{3}{2}})(1p_{\frac{1}{2}})^{-1}}$ has only a small convection current contribution (note that this is a spin-flip transition) so that its squared form factor is primarily just due to the magnetization contribution, and therefore increases rapidly with q. These two squared form factors vary so rapidly that even if these states are mixed with other 1^-, $T=1$ states to form Ψ_2 and Ψ_3, the squared form factors still show the same behavior, i.e., they are dominated by these two basis states. The squared form factor of the giant resonance is therefore essentially the sum of an increasing function and a decreasing function of q, which then has a minimum in the region under consideration.

Calculations similar to the preceding have been applied by one of us (F.H.L.) to the $E1$ form factors in O^{16} with also quite remarkable success and will be discussed in a forthcoming publication.

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APPENDIX A: HYDRODYNAMICAL MODEL

In this model[36] the neutrons and protons are treated as two interpenetrating fluids inside a rigid spherical volume with constant total density,

$$\rho_p(\mathbf{r},t)+\rho_n(\mathbf{r},t)=\rho_0=\rho_p{}^0+\rho_n{}^0=\frac{Z}{A}\rho_0+\frac{N}{A}\rho_0, \quad \text{(A1)}$$

where

$$\tfrac{4}{3}\pi r_0{}^3\rho_0=1, \quad r_0\cong 1.4\times 10^{-13}\ \text{cm}.$$

The interaction between these fluids arises from the nuclear symmetry energy[37,42,43]

$$E_s=K[(N-Z)^2/A] \quad \text{(A2)}$$

via the energy density $\epsilon(\mathbf{r},t)$, which has the static equilibrium value

$$\epsilon_0=K\frac{(\rho_n{}^0-\rho_p{}^0)^2}{\rho_0}, \quad K\cong 24\ \text{MeV}.^{37}$$

Using the definitions

$$\begin{aligned}&\eta(\mathbf{r},t)=\rho_p(\mathbf{r},t)-\rho_p{}^0, \quad \rho_r=\frac{\rho_n{}^0\rho_p{}^0}{\rho_0},\\ &\mathbf{v}(\mathbf{r},t)=\mathbf{v}_p(\mathbf{r},t)-\mathbf{v}_n(\mathbf{r},t) \quad \text{(flow velocities)},\\ &\mathbf{V}=(\rho_0)^{-1}(\rho_n\mathbf{v}_n+\rho_p\mathbf{v}_p),\end{aligned} \quad \text{(A3)}$$

and assuming

$$\mathbf{V}=0 \quad \text{and} \quad \nabla\times\mathbf{v}=0, \quad \text{i.e.,} \quad \mathbf{v}=-\nabla\phi(\mathbf{r},t), \quad \text{(A4)}$$

we have, to first order in η and ϕ, the Lagrangian:

$$L=\int d^3r\left\{\frac{\rho_0M^2}{16K}\left(\frac{\partial}{\partial t}\phi\right)^2-\frac{M}{2}\rho_r(\nabla\phi)^2\right\} \quad \text{(A5)}$$

together with the subsidiary condition (continuity equation)

$$(\partial/\partial t)\eta+\rho_r\nabla\cdot\mathbf{v}=0. \quad \text{(A6)}$$

The resulting equation of motion[44]

$$\left(\frac{\partial^2}{\partial t^2}-\frac{8K}{M}\frac{NZ}{A^2}\nabla^2\right)\eta(\mathbf{r},t)=0 \quad \text{(A7)}$$

together with the boundary condition

$$\frac{\partial}{\partial r}\eta\Big|_{r=R}=0, \quad \text{where} \quad R=r_0A^{1/3} \quad \text{(A8)}$$

lead to the complete set of orthonormal modes for η:

$$\begin{aligned}&\eta_{lmn}(\mathbf{r},t)=e^{i\omega_{ln}t}\psi_{lmn}(\mathbf{r}), \quad \omega_{ln}=k_{ln}u,\\ &\psi_{lmn}(\mathbf{r})=A_{lmn}j_l(k_{ln}r)Y_{lm}(\theta,\phi),\\ &A_{lmn}=\frac{1}{j_l(k_{ln}R)}\left[\tfrac{1}{2}R^3\left(1-\frac{l(l+1)}{k_{ln}{}^2R^2}\right)\right]^{-1/2},\\ &u^2=\frac{8K}{M}\frac{NZ}{A^2},\\ &k_{ln}R \ \text{is the } (n+1)\text{th root of } \frac{d}{dr}j_l(k_{ln}r)\Big|_{r=R}=0,\\ &\qquad n=0,1,2,\cdots,\end{aligned} \quad \text{(A9)}$$

[42] C. F. von Weizsäcker, Z. Physik **96**, 431 (1935).
[43] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952), Chap. VI.
[44] H. Goldstein, *Classical Mechanics* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1959).

where $j_\nu(x)$ are the spherical Bessel functions[24] and Y_{lm} are the spherical harmonics.[14]

We introduce the canonical commutation rules for $\phi(\mathbf{r},t)$ and $\pi(\mathbf{r},t)=-\delta L/\delta(\partial\phi(\mathbf{r},t)/\partial t)$ in the Schrödinger representation;

$$[\phi(\mathbf{r}),\phi(\mathbf{r}')]=[\pi(\mathbf{r}),\pi(\mathbf{r}')]=0,\quad [\phi(\mathbf{r}),\pi(\mathbf{r}')]=i\hbar\delta(\mathbf{r}-\mathbf{r}'). \tag{A10}$$

Expanding

$$\phi(\mathbf{r})=\sum_{lmn} q_{lmn}\psi_{lmn}(\mathbf{r}),\quad \pi(\mathbf{r})=\sum_{lmn} p_{lmn}\psi_{lmn}^*(\mathbf{r}) \tag{A11}$$

and defining

$$q_{lmn}=\left(\frac{\hbar}{2\mu\omega_{ln}}\right)^{1/2}[a_{lmn}+(-1)^m a_{l-mn}^\dagger],$$
$$p_{lmn}=i(\tfrac{1}{2}(\mu\hbar\omega_{ln}))^{1/2}[a_{lmn}^\dagger-(-1)^m a_{l-mn}], \tag{A12}$$

where

$$\mu=\rho_0 M^2/8K,$$

we obtain the Hamiltonian

$$H=\int d^3r\left[\frac{4K}{\rho_0 M^2}\pi^2+\frac{M\rho_r}{2}(\nabla\phi)^2\right]=\sum_{lmn}\hbar\omega_{ln}[a_{lmn}^\dagger a_{lmn}+\tfrac{1}{2}] \tag{A13}$$

together with the commutation rules

$$[a_{lmn},a_{l'm'n'}]=[a_{lmn}^\dagger,a_{l'm'n'}^\dagger]=0,\quad [a_{lmn},a_{l'm'n'}^\dagger]=\delta_{ll'}\delta_{mm'}\delta_{nn'}. \tag{A14}$$

The operator

$$\mathbf{L}=\frac{i}{\hbar}\sum_{lnmm'} p_{lm'n}(lm'|\mathbf{L}_{op}|lm)q_{lmn}, \tag{A15}$$

where

$$(lm'|\mathbf{L}_{op}|lm)=\int\int d\Omega Y_{lm'}^*(\Omega)[-i\mathbf{r}\times\nabla]Y_{lm}(\Omega)$$

satisfies

$$\mathbf{L}\times\mathbf{L}=i\mathbf{L}. \tag{A16}$$

It is *not*, however, the kinetic angular momentum of the system, which is actually zero in this model. The state $a_{lmn}^\dagger|0\rangle$ diagonalizes $\mathbf{L}^2$ and L_Z with eigenvalues $l(l+1)$ and m.

The giant resonance is taken to be the low-lying 1^- state:

$$|1m0)=a_{1m0}^\dagger|0). \tag{A17}$$

This has an energy

$$\hbar\omega_{10}=2.08\frac{\hbar}{R}\left(\frac{8K}{M}\frac{NZ}{A^2}\right)^{1/2}. \tag{A18}$$

The transverse electric form factor of the giant resonance is given by

$$(1^-\|T_1^{el}(q)\|0^+)=\sqrt{3}(0|a_{100}T_{10}^{el}(q)|0),$$

where

$$T_{1M}^{el}(q)=\frac{1}{q}\int d^3r\mathbf{j}_N(\mathbf{r})\cdot\nabla\times j_1(qr)\mathcal{Y}_{111}^M(\theta,\phi)$$

and

$$c\mathbf{j}_N(\mathbf{r})=\rho_p(\mathbf{r})\mathbf{v}_p(\mathbf{r})\cong-\rho_r\nabla\phi(\mathbf{r}). \tag{A19}$$

Integration by parts yields only a surface term since $\nabla\times\mathbf{j}_N(\mathbf{r})=0$. Using the identities[14]

$$\mathcal{Y}_{111}^M(\theta,\phi)=\frac{1}{i\sqrt{2}}(\mathbf{r}\times\nabla)Y_{1M}(\theta,\phi),$$
$$\mathbf{r}\cdot\nabla Y_{1M}(\theta,\phi)=0,\quad\left(\nabla^2+\frac{2}{r^2}\right)Y_{1M}(\theta,\phi)=0, \tag{A20}$$

we obtain

$$T_{1M}^{el}(q)=\frac{R\sqrt{2}}{iqc}j_1(qR)\rho_r\times\int\int d\Omega\phi(\mathbf{r})|_{r=R}Y_{1M}(\theta,\phi), \tag{A21}$$

$$|(1^-\|T_1^{el}(q)\|0^+)|^2=24\frac{\hbar^2}{M^2c^2}\rho_0R^2K\left(\frac{NZ}{A^2}\right)^2\frac{1}{\hbar\omega_{10}}\times\left[\frac{A_{100}}{q}j_1(qR)j_1(k_{10}R)\right]^2. \tag{A22}$$

APPENDIX B: PHOTOEXCITATION

The integrated cross section for photoexcitation of the nucleus into an excited state ϕ_n by a photon $(\mathbf{k},\lambda)$ is given by[45]

$$\int_{\text{abs line}}\sigma(\omega)d\omega=\frac{2\pi}{\hbar^2c}\Omega\langle|(\phi_n|H_1|\phi_0,\mathbf{k}\lambda)|^2\rangle_{\text{av}}, \tag{B1}$$

where Ω is the "volume of the universe" and

$$(\phi_n|H_1|\phi_0,\mathbf{k}\lambda)=\left(\phi_n\Big|\sum_{J=1}^{\infty}\sum_{M=-J}^{J}i^J\times\left[\frac{2\pi e^2\hbar c^2}{2\Omega\omega_k}(2J+1)\right]^{1/2}\{T_{JM}^{el}(k)+\lambda T_{JM}^{mag}(k)\}\times\mathfrak{D}_{M\lambda}^J(-\phi_k,-\theta_k,\phi_k)\Big|\phi_0\right), \tag{B2}$$

[45] W. Heitler, *The Quantum Theory of Radiation* (Clarendon Press, Oxford, England, 1954), 3rd ed.; see also Ref. 43, Appendix B.

where $D_{M\lambda}^{J}$ are the rotation matrices defined in Ref. 14. If the initial and final states are both eigenstates of $\mathbf{J}^2$ and parity, then

$$|(J_f\pi_f\|T_J^{\mathrm{el}}(q)+\lambda T_J^{\mathrm{mag}}(q)\|J_i\pi_i)|^2 = |(J_f\pi_f\|T_J^{\mathrm{el}}(q)\|J_i\pi_i)|^2 + |(J_f\pi_f\|T_J^{\mathrm{mag}}(q)\|J_i\pi_i)|^2 \quad \text{(B3)}$$

since one or the other term must vanish.

Using the techniques in Edmonds (Ref. 14) to perform the sum over final nuclear orientations and average over initial nuclear orientations and photon polarizations λ we find

$$\langle|(\phi_n|H_1|\phi_0,\mathbf{k}\lambda)|^2\rangle_{\mathrm{av}} = \sum_{J=1}^{\infty}\frac{2\pi e^2\hbar c^2}{2\Omega\omega_k}\frac{1}{2J_i+1} \times\{|(J_f\|T_J^{\mathrm{el}}(k)\|J_i)|^2+|(J_f\|T_J^{\mathrm{mag}}(k)\|J_i)|^2\}. \quad \text{(B4)}$$

Hence, we obtain the formula (4) given in Sec. II:

$$\int\sigma(E)dE=(2\pi)^3\frac{\alpha(\hbar c)^2}{E_{fi}}\frac{1}{2J_i+1} \times\sum_J\{|(J_f\|T_J^{\mathrm{el}}(k)\|J_i)|^2 + |(J_f\|T_J^{\mathrm{mag}}(k)\|J_i)|^2\}, \quad \text{(B5)}$$

where $E_{fi}=\hbar ck$ and $\alpha=e^2/4\pi\hbar c$ is the fine-structure constant.